# REGS REPORT: THE MORAVA $E$-THEORY OF EILENBERG-MAC LANE SPACES

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## 1. Introduction

Chromatic homotopy, the confluence of stable homotopy and the geometry of formal groups, uses the following dictionary to interpret some of its basic tools:

| Spectrum | Name | Geometric counterpart |
|---|---|---|
| $MU$ | complex cobordism | moduli stack of formal groups, |
| $BP$ | Brown-Peterson theory | moduli stack of $p$-typical formal groups, |
| $E(n)$ | Johnson-Wilson theory | open substack of formal groups of height at most $n$, |
| $K(n)$ | Morava $K$-theory | geometric point of this stack, |
| $E_n$ | Morava $E$-theory | universal deformation of this point. |

The homology theory $K(n)_*$ is particularly popular among those interested in computations, in large part because $\pi_* K(n)$ is a field and because it comes with a Künneth isomorphism:

$$K(n)_* X \times Y = K(n)_* X \otimes_{\pi_* K(n)} K(n)_* Y.$$

These properties fuel many arguments involving the decomposition of spaces, where more complex homology theories would instead have nontrivial spectral sequences obstructing similar results.

In the 1960s, Ravenel and Wilson [8] computed $K(n)_* K(\mathbb{Z}, q)$ for all $n$ and $q$, using the Künneth isomorphism to show that the functor $K(n)_*$ took the ring object $K(\mathbb{Z}, *)$ to a $\pi_* K(n)$-coalgebraic ring $K(n)_* K(\mathbb{Z}, *)$ and then analyzing the addition and multiplication maps. Our primary goal this summer was to compute the Morava $E$-theory of this same ring $K(\mathbb{Z}, *)$ by studying Lubin and Tate's construction [6] of the universal deformation of the Honda formal group spf $K(n)^* BU(1)$ and interleaving it with Ravenel and Wilson's work.

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## 2. Past work

The entire rest of this paper rests on the argument of Ravenel and Wilson [8]; the first step in the project was to interpret their work, which we will sketch here. Suppose $E$ and $F$ are both ring spectra satisfying

$$E_*(\Omega^{\infty-n} F \times \Omega^{\infty-m} F) \cong E_* \Omega^{\infty-n} F \otimes E_* \Omega^{\infty-m} F.$$

Then $E_* \Omega^{\infty-*} F$ forms a $\pi_* E$-coalgebraic ring, sometimes called a "Hopf ring." This means that we have maps

$$E_*(\Omega^{\infty-n} F) \otimes E_*(\Omega^{\infty-n} F) \xrightarrow{*} E_* \Omega^{\infty-n} F,$$

$$E_*(\Omega^{\infty-n} F) \otimes E_*(\Omega^{\infty-m} F) \xrightarrow{\circ} E_* \Omega^{\infty-n-m} F$$

induced from the unstable addition and multiplication maps on the ring spectrum $F$. The existence of so many operations allows us to describe large chunks of a Hopf ring using just a few elements, a useful calculational technique.

Thanks to Morava $K$-theory's Künneth isomorphism, we can set $E = K(n)$ and $F = HG$ in the above. Ravenel and Wilson calculate that $K(n)_*K(\mathbb{Z}/p^j, *)$ is the free $K(n)_*$-Hopf ring on $K(n)_*K(\mathbb{Z}/p^j, 1)$, which is in turn calculated as the $K(n)$-homology of the homotopy fiber of the map

$$\mathbb{C}P^\infty \xrightarrow{p^j} \mathbb{C}P^\infty.$$

This is dual to the $j$th iterate of the $p$-series in cohomology, which is well-understood by construction; Morava $K$-theory is designed to support the equality $[p]_{K(n)}(x) = v_n x^{p^n}$ over the coefficient ring $\pi_*K(n) = \mathbb{F}_p[v_n^{\pm 1}]$. Therefore, up to a unit we have

$$[p^j]_{K(n)}(x) = x^{p^{jn}}.$$

Analyzing this map yields the following calculation: $K(n)_*K(\mathbb{Z}/p^j, 1)$ is a free $\pi_*K(n)$-module on elements $a_i \in K(n)_{2i}K(\mathbb{Z}/p^j, 1)$ for $0 \le i < p^{nj}$, with coproduct

$$\psi(a_i) = \sum_{k=0}^{i} a_k \otimes a_{i-k}.$$

The algebra structure is generated by the elements $a_{(i)} = a_{p^i}$, subject to the relation

$$a_{(n+i-1)}^{*p} = a_{(i)},$$

up to a unit.

The statement that $K(n)_*K(\mathbb{Z}/p^j, *)$ is the free Hopf ring on $K(n)_*K(\mathbb{Z}/p^j, 1)$ can be interpreted one degree at a time to recover the calculation of $K(n)_*K(\mathbb{Z}/p^j, q)$ for individual $q$. As notation, for $I$ be a multi-index of length $q$ consisting of nonnegative integers less than $nj$, we write

$$a_I = a_{(I_1)} \circ \cdots \circ a_{(I_q)}.$$

Because of the freeness, we have the relations

$$a_{(i)} \circ a_{(k)} = -a_{(k)} \circ a_{(i)},$$
$$a_{(i)} \circ a_{(i)} = 0,$$
$$a_{(i)} \circ a_{(k)} = 0 \text{ if } i < n \text{ and } k < n(j-1),$$
$$a_{(i)} \circ a_{(k)} = a_{(i-n)} \circ a_{(k+n)} \text{ up to a unit, if } n \le i \text{ and } k < n(j-1).$$

It follows that any $a_I$ can be rearranged so that $I$ is increasing.

Then, the reduction $\mathbb{Z}/p^{j+1} \to \mathbb{Z}/p^j$ gives rise to a map $r_j : K(n)_*K(\mathbb{Z}/p^{j+1}, *) \to K(n)_*K(\mathbb{Z}/p^j, *)$, which, up to a unit, acts by

$$r_j(a_{(I_1, \ldots, I_q)}) = a_{(I_1 - n, \ldots, I_q - n)}.$$

The inclusion $\mathbb{Z}/p^j \to \mathbb{Z}/p^{j+1}$ acts up to a unit by

$$i_j(a_{(I_1, \ldots, I_q)}) = a_{(I_1, I_2 + n, \ldots, I_q + n)}.$$

As a corollary, since we are working $p$-locally, we can calculate $K(n)_*K(\mathbb{Z}, q) \cong K(n)_*K(\mathbb{Z}/p^\infty, q)$ by inverse limit.

We also give a formal group interpretation of this result, following Ando and Strickland [2]. Make note of the following two pieces of notation:

$$X^E = \operatorname{spec} E_*X,$$
$$X_E = \operatorname{spf} E^*X = \operatorname*{colim}_{\substack{Y \subseteq X \\ Y \text{ finite}}} \operatorname{spec} E^*Y.$$

Fix a finite abelian group $A$ and write $A^\vee = \underline{\operatorname{Hom}}(A, \mathbb{C}^\times)$ for its Pontryagin dual. Then, the various characters corresponding to the elements of $A$ give rise to a map

$$\coprod_{a \in A} BA \xrightarrow{\amalg B\chi} B\mathbb{C}^\times,$$

and applying the functor $-_E$ for a complex-oriented cohomology theory $E$ gives a map

$$A^\vee \times BA_E \cong \left(\coprod_{a \in A} BA\right)_E \to \mathbb{C}P_E^\infty,$$

where $A^\vee$ denotes the associated constant group scheme. This map is an element of $\underline{\mathrm{Hom}}(A^\vee, \mathbb{C}P_E^\infty)$ of groups over $BA_E$, so is represented by a section $BA_E \to \underline{\mathrm{Hom}}(A^\vee, \mathbb{C}P_E^\infty)$; the first result of Ravenel and Wilson is that this is an isomorphism for $A = \mathbb{Z}/p^j$ and $E = K(n)$. Specializing to this case, we invert this isomorphism to get a map $\mathbb{C}P_{K(n)}^\infty[p^j] := \underline{\mathrm{Hom}}(\mathbb{Z}/p^j, \mathbb{C}P_{K(n)}^\infty) \to B\mathbb{Z}/p_{K(n)}^j$. That $K(\mathbb{Z}/p^j, *)_{K(n)}$ is a ring object means we get a map out of the free object produced by $\Lambda^*$:

$$\Lambda^* \mathbb{C}P_{K(n)}^\infty[p^j] \to K(\mathbb{Z}/p^j, *)_{K(n)}.$$

This is the map we'll study.

Ravenel and Wilson show that this map is an isomorphism, i.e., $K(\mathbb{Z}/p^j, q)_{K(n)}$ is the universal example of a scheme equipped with an alternating map from $\mathbb{C}P_{K(n)}^\infty[p^j]^q$. The freeness result above exactly means that $K(\mathbb{Z}/p^j, *)_{K(n)}$ is the free graded-commutative formal ring scheme over $\mathbb{Z}/p$ generated by $K(\mathbb{Z}/p^j, 1)_{K(n)}$, which we've calculated to be $\mathbb{C}P_{K(n)}^\infty[p^j]$. We may define free objects in this setting in the obvious way: since we have colimits and tensor products of formal schemes, we can write

$$\Lambda^q \mathbb{C}P_{K(n)}^\infty[p^j] = \mathbb{C}P_{K(n)}^\infty[p^j]^{\otimes q}/\Sigma_q,$$

where $\Sigma_q$ acts with signs, and these constitute the homogenous degree pieces of the formal ring scheme.

Now, we use Cartier duality to write

$$K(\mathbb{Z}/p^j, q)^{K(n)} = \underline{\mathrm{Hom}}(K(\mathbb{Z}/p^j, q)_{K(n)}, \mathbb{G}_m).$$

Since the augmentation ideal in $K(n)^* K(\mathbb{Z}/p^j, q)$ is topologically nilpotent, we know that $K(\mathbb{Z}/p^j, q)_{K(n)}$ is a connected formal neighborhood of the identity element, and hence any map $K(\mathbb{Z}/p^j, q)_{K(n)} \to \mathbb{G}_m$ which sends the identity to zero must factor as

$$K(\mathbb{Z}/p^j, q)_{K(n)} \to \widehat{\mathbb{G}_m} \to \mathbb{G}_m.$$

We may then identify $K(\mathbb{Z}/p^j, q)^{K(n)}$ with the scheme

$$W(n, q) \subseteq \underline{\mathrm{Hom}}(\mathbb{C}P_{K(n)}^\infty[p^j]^q, \widehat{\mathbb{G}_m})$$
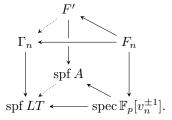
of $q$-variate alternating, rigid maps.

## 3. This summer's work

Now we turn to $(E_n)_* K(\mathbb{Z}, *)$. The work of Lubin and Tate [6] describes the deformation theory of the Honda formal group $F_n = \mathbb{C}P_{K(n)}^\infty$ of height $n$ over $\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}]$. They construct a formal group $\Gamma_n$ over the ring

$$LT = \pi_* E_n = \mathbb{Z}_{(p)}[\![v_1, \ldots, v_{n-1}]\!][v_n^{\pm 1}]$$

so that $\Gamma_n$ is terminal in the category of deformations of $F_n$ to complete, local rings in the sense that the dashed arrow in the following diagram of pullback squares always exists uniquely:

$$
\begin{array}{ccc}
 & F' & \\
\Gamma_n \longleftarrow & | & F_n \\
 & \mathrm{spf}\, A & \\
\mathrm{spf}\, LT \longleftarrow & & \mathrm{spec}\, \mathbb{F}_p[v_n^{\pm 1}].
\end{array}
$$

This formal group $\Gamma_n$ is the formal group associated to the Morava $E$-theory, $\mathbb{C}P_{E(n)}^\infty$.

It is difficult to explicitly compute the full structure of $\Gamma_n$. Lubin and Tate's approach to the problem is to study deformations of formal groups $G_0$ over schemes spf $R_0$ to formal groups $G$ over schemes spf $R$ along continuous maps $R \twoheadrightarrow R_0$ whose kernel is a square-zero ideal in $R$. The powers of $\mathfrak{m} = \langle p, v_1, \ldots, v_{n-1} \rangle$ in $LT$ give rise to a sequence

$$LT \twoheadrightarrow \cdots \twoheadrightarrow LT/\mathfrak{m}^{r+1} \twoheadrightarrow LT/\mathfrak{m}^r \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{F}_p[v_n^{\pm 1}].$$

Each of these connecting maps has as its kernel a square-zero maximal ideal, allowing Lubin and Tate to study the deformations along the individual steps in the sequence.

We will mimic this approach, deforming Ravenel and Wilson's argument along with the Honda formal group law. Fundamental parts of their proof require that we work in characteristic $p$, so we will instead work with the sequence

$$LT \twoheadrightarrow LT/p \twoheadrightarrow \cdots \twoheadrightarrow LT/\langle p, \mathfrak{m}^{r+1}\rangle \twoheadrightarrow LT/\langle p, \mathfrak{m}^r\rangle \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{F}_p[v_n^{\pm 1}].$$

This sequence is inherited from the level of spectra by a sequence of maps of complex-oriented, structured ring spectra

$$E_n \to E_n/p =: E \to \cdots \to E/\mathfrak{m}^{r+1} \to E/\mathfrak{m}^r \to \cdots \to K(n).$$

The first map $E_n \to E_n/p$ we will deal with separately; the remainder fits the general template set out by Lubin and Tate.

Recall the following general facts about a complex oriented spectrum $R$. The action of $R$ on $\mathbb{C}P^\infty$ is key: $R^*\mathbb{C}P^\infty$ and $R_*\mathbb{C}P^\infty$ are dual Hopf algebras, and we have an isomorphism $R^*\mathbb{C}P^\infty \cong R^*[\![x]\!]$, where $x$ is a certain class in degree 2. The comultiplication on $R^*\mathbb{C}P^\infty$ produces a bivariate power series $\Delta(x)$, which we notate as $x +_R y$. Dually, $R_*\mathbb{C}P^\infty$ is a free $R_*$-module on generators $\beta_i \in R_{2i}\mathbb{C}P^\infty$. Writing $\beta(s)$ for $\sum_i \beta_i s^i$, duality of Hopf algebras determines its algebra structure to be given by

$$\beta(s) * \beta(t) = \beta(s +_R t).$$

We also have a coalgebra structure on $R_*\mathbb{C}P^\infty$ determined by

$$\psi(\beta_n) = \sum_{i=0}^{n} \beta_i \otimes \beta_{n-i}.$$

In the remainder of this section, each of the following theorems will be assumed true for the spectrum $E/\mathfrak{m}^{r-1}$; we will demonstrate them in turn for $E/\mathfrak{m}^r$. We will implicitly identify the elements of $E/\mathfrak{m}^{r-1}$ with the lexically equivalent elements of $E/\mathfrak{m}^r$. The module $M$ is then spanned by monomials of the form $(v_1, \ldots, v_{n-1})^I$ where $I$ is a multi-index of length $(n-1)$ and weight $(r-1)$.

**Note:** To get a handle on this lifted computation, our first goal was to work out the case $r = 2$. We've left several of those computations in footnotes to the text, although the reader should be aware that they were done under the assumption that the follow equation held:

$$[p]_{E/\mathfrak{m}^2}(x) = \sum_{i=1}^{n} v_i x^{p^i}.$$

This has since come under doubt.

**3.1.** $(E/\mathfrak{m}^r)_* B\mathbb{Z}/p^j$. To computationally demonstrate that $(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p^j, *)$ is the free Hopf ring on $(E/\mathfrak{m}^r)_* B\mathbb{Z}/p^j$, we must first describe this generating algebra.

**Theorem 1:** For $k$ not a power of $p$, $\beta_k \in (E/\mathfrak{m}^r)_* \mathbb{C}P^\infty$ is decomposable with respect to the $*$-product.

*Proof.* This is essentially identical to Ravenel and Wilson [8, Theorem 5.6]. $\square$

It will therefore suffice to investigate the algebra generators $\beta_{(k)} = \beta_{p^k}$. The case of Morava $K$-theory corresponds to a previous result [8, Theorem 5.6], which states that the governing relation is of the form

$$\beta_{(n+k-1)}^{*p} = v_n^{p^k} \beta_{(k)}.$$

**Theorem 2:** Write $b'_{k,I}$ for the element of $K(n)_*\mathbb{C}P^\infty$ satisfying

$$\beta^{*p}_{(k)} = \sum_{\substack{|I| < r-1 \\ \ell(I)=n-1}} (v_1, \ldots, v_{n-1})^I b'_{k,I}$$

with $\beta_{(k)}$ considered as an element of $(E/\mathfrak{m}^{r-1})_*\mathbb{C}P^\infty$, and $b_{k,I}$ for the coefficients of the summands in

$$\beta^{*p}_{(k)} = \sum_{\substack{|I| \leq r-1 \\ \ell(I)=n-1}} (v_1, \ldots, v_{n-1})^I b_{k,I}$$

with $\beta_{(k)}$ considered as an element of $(E/\mathfrak{m}^r)_*\mathbb{C}P^\infty$. Then $b'_{k,I} = b_{k,I}$ for all $I$ with $|I| < r - 1$.[1]

*Proof sketch.* This follows from expanding the equality $\beta([p]_R(s)) = \beta(s)^{*p}$. That we work in characteristic $p$ is important for manipulating this expression. When $p = 0$, we can rewrite the right-hand side as

$$\beta(s)^{*p} = \sum_{i=0}^\infty \sum_{\substack{|I|=i \\ \ell(I)=p}} \{\# \text{ of ways to reorder } I\}\beta_{*I}s^i \equiv \sum_{i=0}^\infty \beta^{*p}_i s^{ip} \quad (\text{mod } p). \qquad \square$$

The space $B\mathbb{Z}/p^j$ appears in the fiber sequence

$$B\mathbb{Z}/p^j \xrightarrow{\delta} \mathbb{C}P^\infty \xrightarrow{p^j} \mathbb{C}P^\infty.$$

Converting the fiber itself to a fibration gives a fiber sequence

$$S^1 \to B\mathbb{Z}/p^j \xrightarrow{\delta} \mathbb{C}P^\infty.$$

Because $\pi_1\mathbb{C}P^\infty = 0$, the fibration $\delta$ is an oriented spherical fibration, which yields a Gysin sequence of the form

$$
\begin{array}{ccc}
(E/\mathfrak{m}^r)_*B\mathbb{Z}/p^j & \xrightarrow{\quad\delta_*\quad} & (E/\mathfrak{m}^r)_*\mathbb{C}P^\infty \\
& \overset{\partial}{\nwarrow} \qquad\qquad \overset{\Phi}{\swarrow} & \\
& (E/\mathfrak{m}^r)_{*-2}\mathbb{C}P^\infty, &
\end{array}
$$

where the action of $\Phi$ is described by $\Phi(y) = [p^j]_{\Gamma_n}(x) \frown y$.

**Theorem 3:** The Hopf algebra $(E/\mathfrak{m}^r)_*B\mathbb{Z}/p^j$ occurs as the kernel of $\Phi$, a subalgebra of $(E/\mathfrak{m}^r)_*\mathbb{C}P^\infty$. As an $(E/\mathfrak{m}^r)_*$-module, it is free on generators $a_i$, $0 \leq i < p^{nj}$, reducing to $\beta_i$ in $K(n)_*B\mathbb{Z}/p^j$.[2]

*Proof sketch.* We show that $\Phi$ is surjective by producing classes $y_k$ so that $\Phi(y_k) = \beta_k$; this follows from the previous theorem by using the analogous result in $E/\mathfrak{m}^{r-1}$, then using surjectivity in the case of $E/\mathfrak{m} \cong K(n)$ to correct the classes sitting in $\mathfrak{m}^{r-1}/\mathfrak{m}^r$. Since $\Phi$ is surjective, $\partial$ vanishes and $\delta_*$ is injective — moreover, as $\delta$ is a map of $H$-spaces, $\delta_*$ is an inclusion of Hopf algebras. The $E/\mathfrak{m}^{r-1}$-homology of $B\mathbb{Z}/p^j$ is generated by elements $a'_k$ for $k < nj$ which restrict to $\beta_k$ on $K(n)$-homology; we can apply the same corrective procedure to produce elements $a_k$ which span the kernel in $E/\mathfrak{m}^r$-homology and restrict to $a'_k$. $\qquad \square$

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[1] In the case of $r = 2$, we compute $\beta^{*p}_{(k)} = v_n^{p^{k+1-n}}\beta_{(k+1-n)}$ for $k + 1 \geq n$ and $\beta^{*p}_{(k)} = v_{k+1}\beta_{(0)}$ for $k + 1 < n$.

[2] Set $y_m = v_n^{-\sum_{i=0}^{j-1} p^{in}}\beta_{m+p^{jn}} - \sum_{i=1}^{n-1} v_i v_n^{p^{i(j-1)+n\frac{(j-1)(j-2)}{2}}-2\sum_{i=0}^{j-1} p^{in}}\beta_{m+2p^{jn}-i-p^{(j-1)n}}$; then $\Phi(y_m) = \beta_m$. Hence, the kernel is spanned by elements $a_k = \beta_k$ for $k \leq p^{(j-1)n}$ and $a_k = \beta_k - \sum_{i=1}^{n-1} v_i v_n^{\kappa(i,j,n)}\beta_{k-i-p^{(j-1)n}+p^{jn}}$ for $p^{(j-1)n} < k < p^{jn}$ and $\kappa(i,j,n) = p^{i(j-1)+n\frac{(j-1)(j-2)}{2}} - \sum_{m=0}^{j-1} p^{mn}$. These $a_k$ satisfy the same $-^{*p}$ relations as the $\beta_k$.

We will also need to understand how $(E/\mathfrak{m}^r)_* B\mathbb{Z}/p^j$ relate for various $j$. We have a diagram of short exact sequences

$$
\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{p^{j+1}} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p^{j+1} \\
{\scriptstyle p}\downarrow & & \| & & \downarrow \\
\mathbb{Z} & \xrightarrow{p^j} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p^j.
\end{array}
$$

By delooping a bit, we have an induced diagram of circle bundles

$$
\begin{array}{ccccc}
S^1 & \longrightarrow & B\mathbb{Z}/p^{j+1} & \longrightarrow & \mathbb{C}P^\infty \\
\| & & {\scriptstyle r_j}\downarrow & & {\scriptstyle p}\downarrow \\
S^1 & \longrightarrow & B\mathbb{Z}/p^j & \longrightarrow & \mathbb{C}P^\infty,
\end{array}
$$

and hence computing $(E/\mathfrak{m}^r)_* r_j$ reduces to understanding the action of $-^{*p}$ on $(E/\mathfrak{m}^r)_* \mathbb{C}P^\infty$, as above.

**3.2.** $(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p^j, q)$ **for** $q > 1$**.**

*3.2.1. The bar spectral sequence.* Let $G$ be an H-group and $BG$ the bar complex model for its classifying space:

$$
BG = \left( \coprod_{i \geq 0} G^i \times \Delta^i \right) / \sim,
$$

where $\sim$ is some pushout identification gluing together faces of the simplices $\Delta^i$. This model has an evident filtration

$$
B_k G = \left( \coprod_{0 \leq i \leq k} G^i \times \Delta^i \right) / \sim,
$$

$$
\operatorname{colim}(B_0 G \hookrightarrow B_1 G \hookrightarrow \cdots \hookrightarrow B_k G \hookrightarrow \cdots) \simeq BG
$$

with filtration quotients described by $B_k G / B_{k-1} G \simeq \Sigma^k G^{\wedge k}$. For an arbitrary multiplicative cohomology theory $h$ satisfying $h_* K(G, q)^{\wedge k} \cong h_* K(G, q)^{\otimes k}$, this filtration gives rise to a spectral sequence with $E^1$ page $E^1_{k,*} = h_* K(G, q)^{\otimes k}$ — moreover, $(E_1, d_1)$ is the bar resolution of $h_*$ in the category of $h_* K(G, q)$-comodules, and hence

$$
E^2_{*,*} \cong H_{*,*} h_* K(G, q) := \operatorname{Tor}^{h_* K(G,q)}_{*,*}(h_*, h_*).
$$

The cup product map respects this filtration in the sense that the dashed arrow in the following diagram exists (see Ravenel and Wilson [8, Theorem 1.9] for reference):

$$
\begin{array}{ccc}
B_k K(G, q) \wedge K(G, q') & \dashrightarrow & B_k K(G, q + q') \\
\downarrow & & \downarrow \\
K(G, q+1) \wedge K(G, q') & \xrightarrow{\ \circ\ } & K(G, q + q' + 1).
\end{array}
$$

The compatibility of the cup product and the filtration tells us that this spectral sequence is compatible with the cup product as well; we have maps

$$
\circ : E^s_{*,*} K(G, q) \otimes_{h_*} h_* K(G, q') \to E^s_{*,*} K(G, q + q')
$$

converging to the cup product map on the $E^\infty$ page and interacting with the differentials via $d^r(x \circ y) = d^r(x) \circ y$. This formula gives us an immense amount of control over the parts of the spectral sequence that can be described using the $\circ$-product.

To illustrate how this spectral sequence works and to provide an inductive foothold, we will describe

$$
H_{*,*}(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p^j, 0) \Rightarrow (E/\mathfrak{m}^r)_* B\mathbb{Z}/p^j.
$$

The space $K(\mathbb{Z}/p^j, 0)$ is discrete, and so its homology is easy to compute: it's the free $(E/\mathfrak{m}^r)_*$-module on that set. The ring structure of $\mathbb{Z}/p^j$ gives us a ring-ring structure

$$(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p^j, 0) \cong (E/\mathfrak{m}^r)_*[\mathbb{Z}/p^j].$$

This is a truncated polynomial algebra under the $*$-product, generated by the homology class $[1] - [0]$. The homology of a truncated polynomial algebra is a well known calculation; we write $x$ for $[1] - [0]$ in

$$H_{*,*}(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p^j, 0) \cong \Lambda[(x)] \otimes \Gamma[(x^{(p-1)p^{j-1}} \mid x^{p^{j-1}})].$$

We have already computed the Hopf algebra $(E/\mathfrak{m}^r)_* B\mathbb{Z}/p^j$, and so we understand both the $E^2$ and $E^\infty$ pages in our spectral sequence; this gives us enough information to calculate what the differentials must be, as for Ravenel and Wilson [8, Lemma 6.10]. In a bar spectral sequence with $E^2$ page of the form $E^2 \cong \bigotimes_{i \in I} \Lambda[x_i] \otimes \Gamma[y_i]$, the only possible nonvanishing differentials are of the form $d^{2p^{n_i}-1} y_i^{[p^{n_i}]} = ax_j$, inducing $dy_i^{[k]} = ax_j y_i^{[k-p^{n_i}]}$. If these generate all the differentials, then the $E^\infty$ page is generated by the classes $y_i^{[p^j]}$, $0 \le j < n_i$; again, see Ravenel and Wilson [8, Lemma 6.9].

Continuing to write $x$ for $[1] - [0]$ in $(E/\mathfrak{m}^r)_*[\mathbb{Z}/p^j]$, this determines the only differential in our spectral sequence to be

$$d^{2p^{nj}-1}(x^{p-1} \mid x)^{[p^{nj}]} = c(x)$$

for some $c \in (\pi_* E/\mathfrak{m}^r)^\times$. This means that the ideal generated by $(x)$ and $(x^{p-1} \mid x)^{[p^k]}$ for $k \ge nj$ vanishes, and what's left is generated by $y^{[p^k]}$ for $k < nj$.

A priori, the interaction of the bar spectral sequence with the multiplicative structure of the target is potentially nontrivial — and in our case this is so. In particular, the product of elements on the $E^\infty$ page may appear in a lower filtration degree than the product of the individual filtration degrees. Figuring this out is referred to as the multiplicative extension problem. In our case, we've seen that the classes $y^{[p^{nj}+r]}$ vanish for $r \ge 0$, but this doesn't mean that the class represented by $(y^{[p^{(n-1)j}]})$ on the $E^\infty$ page has vanishing $p$th $*$-power. Indeed, our previous analysis of $(E/\mathfrak{m}^r)_* B\mathbb{Z}/p^j$ has already produced representatives for $a_{(k)}^{*p}$; these solve the multiplicative extension problem in our spectral sequence.

*3.2.2. The first inductive step.* Now we want to produce a result for $(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p^j, q)$ by induction on $q$ and $j$, but the complexity of the argument necessitates that we first restrict to $j = 1$ and induct on $q$ alone, leaving $j$ for later. Even with this reduction, the computation with the bar spectral sequence is quite involved and our inductive hypothesis quite lengthy. We will require all of the following hypotheses of Theorem 4:

**Theorem 4:** The structure of the bar spectral sequence

$$H_{*,*}(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q-1) \Rightarrow (E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q)$$

is described as follows:

(1) For $I$ a multi-index of length $q - 1$, let $a_I$ denote $a_{I_1} \circ \cdots \circ a_{I_{q-1}}$ (for $I = ()$, set $a_{()} = [1] - [0]$). The $E^2$ page itself is computed to be

$$E^2_{*,*} \cong \left( \bigotimes_{0 < I_1 < \cdots < I_{q-1} \le n-1} \Lambda[(a_I)] \right) \otimes \left( \bigotimes_{0 \le I_1 < \cdots < I_{q-1} < n-1} \Gamma[\gamma_I] \right),$$

where $\gamma_I$ reduces to $(a_I^{*(p-1)} \mid a_I)$ in $H_{*,*} K(n)_* K(\mathbb{Z}/p, q-1)$.[3]

(2) Let $0 \le I_1 < I_2 < \cdots < I_q < n$, and write $\tilde{I} = (I_2, \ldots, I_q)$. Then $a_I$ is represented in $E^\infty_{*,*}$ by $\gamma_{\tilde{I}-I_1-1}^{[p^{I_1}]}$, modulo decomposables.

(3) Let $0 \le I_1 < I_2 < \cdots < I_{q-1} < n-1$ and set $J = (I_2 - I_1 - 1, \ldots, I_{q-1} - I_1 - 1, n - 2 - I_1)$. The differentials in the spectral sequence are determined by

$$d^{2p^{i_1+1}-1} \gamma_J^{[p^{i_1+1}]} = r_I(a_{I+1}),$$

---

[3]In the case $r = 2$, we computed $\gamma_I = (a_I^{*(p-1)} \mid a_I) - v_{I_{q-1}+1} v_n^{-1}(a_{(I_1,\ldots,I_{q-2},n-1)}^{*(p-1)} \mid a_{(I_1,\ldots,I_{q-2},n-1)})$.

for various units $r_I$.

The Hopf algebra structure of $(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q)$ is described as follows:

(1) As an algebra, $(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q)$ is described as the free $(\pi_* E/\mathfrak{m}^r)$-algebra on the $a_I$, modulo the action of the Frobenius.[4]

(2) $(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q)$ is free as an $(\pi_* E/\mathfrak{m}^r)$-module.

(3) The map $F(a_I) = a_I^{*p}$ acts as a Frobenius, with Verschiebung determined by $V(a_I) = a_{I-1}$ whenever $a_{I-1}$ makes sense and $V(a_I) = 0$ otherwise.

The work above demonstrates Theorem 4 for $q = 1$. Selecting a $q > 1$ and assuming the above are true when $q - 1$ is substituted for $q$, we will outline a partial proof for each of these in turn in this section.

**Theorem 5:** The bar spectral sequence $H_{*,*}(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q - 1) \Rightarrow (E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q)$ exists.

*Proof.* In Morava $E$-theory, we have a Künneth spectral sequence of type

$$\mathrm{Tor}_{*,*}^{\pi_* E/\mathfrak{m}^r}((E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q)^{\wedge s}, (E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q)^{\wedge t}) \Rightarrow (E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q)^{\wedge(s+t)}.$$

This is constructed for an $\mathbb{S}$-algebra spectrum by Elmendorf, Kriz, Mandell, and May [3, Ch. IV, Thm. 4.7]. The derived parts of Tor vanish in the presence of free modules, so our assumption that $(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q-1)$ is an even-concentrated, free $(E/\mathfrak{m}^r)_*$-module shows that the spectral sequence collapses, which yields the desired isomorphism. This ensures the existence and convergence of the bar spectral sequence. □

Tate [9] specifies a method for attaching to any ideal $I$ of a Noetherian ring $A$ a differential graded-commutative $A$-algebra which is free as an $A$-module and has $H_* A = A/I$ concentrated in degree 0. In our case, we have $A = (E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q - 1)$, which is finite and hence Noetherian, and

$$I = \langle a_I \mid 0 \leq I_1 < \cdots < I_{q-1} < n \rangle,$$

which has $A/I = \pi_* E/\mathfrak{m}^r$. We begin by introducing for each $0 \leq I_1 < \cdots < I_{q-1} < n$ an exterior class $e_I$ of degree 1 with $de_I = a_I$. From these 1-chains, we can build two kinds of 1-cycles. First, the relator $a_I * a_J = a_J * a_I$ gives rise to the cycle $e_I a_J - a_I e_J$. This is easy to deal with; we calculate

$$d(e_I e_J) = (de_I)e_J + (-1)^{|e_I||e_J|} e_I (de_J)$$

$$= a_I e_J - e_I a_J.$$

Second, we have a complicated expression for $a_I^{*p}$ in terms of lower $*$-powers of other $a_{I'}$; this gives rise to a 1-cycle by taking the difference and replacing one $a_{I'}$ with an $e_{I'}$ in each summand.[5] This is a genuine obstruction to the exactness of $d$, so we introduce divided power generators $f_I$ so that $df_I^{[1]}$ hits this 1-cycle. This completes our resolution.

To compute $\mathrm{Tor}_{*,*}^{(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q-1)}(\pi_* E/\mathfrak{m}^r, \pi_* E/\mathfrak{m}^r)$, we tensor this resolution with $\pi_* E/\mathfrak{m}^r$, considered as an $(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q - 1)$-algebra. This has the effect of deleting the elements $a_I$ from our resolution, and our differentials become

$$de_I = a_I \equiv 0, \text{ and } df_I^{[1]} \equiv -\sum_J e_J c_{I,J}.$$

for various coefficients $c_{I,J} \in \pi_* E/\mathfrak{m}^r$.[6]

In the Morava $K$-theory case when $r = 1$, we have $a_{(n-1)}^{*p} = v_n a_{(0)}$, and we immediately see that all the elements of the form $e_{(0, I_2, \ldots, I_q)}$ and $f_{(I_1, \ldots, I_{q-1}, n-1)}^{[1]}$ vanish when we take homology of this chain complex, as they're connected by a differential of the form

$$df_{(I_1, \ldots, I_{q-1}, n-1)} = (-1)^q v_n e_{(0, I_1, \ldots, I_{q-1})}.$$

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[4]When $r = 2$, $a_I^{*p} = v_{I_q+1}(-1)^{q-1} a_{(0, I_1+1, \ldots, I_{q-1}+1)}$, and hence $(E/\mathfrak{m}^2)_* K(\mathbb{Z}/p, q)$ is generated by $a_I$ as an $\pi_* E/\mathfrak{m}^2$-module, modulo $\langle a_I - v_{I_q+1}(-1)^{q-1} a_{(0, I_1+1, \ldots, I_{q-1}+1)} \rangle$. Something like this will still be true; the Frobenius really just acts on the first component and shifts the others, so it boils down to whatever the Frobenius looks like on $(E/\mathfrak{m}^r)_* B\mathbb{Z}/p$.

[5]When $r = 2$, these cycles look like $a_I^{*(p-1)} e_I + (-1)^{q-1} v_{I_{q-1}+1} e_{(0, I_1+1, \ldots, I_{q-2}+1)}$.

[6]When $r = 2$, we have $df_I^{[1]} = (-1)^{q-1} v_{I_{q-1}+1} e_{(0, I_1+1, \ldots, I_{q-2}+1)}$.

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Something similar must occur for $r > 1$, but here we begin to run into problems with our poor control on the $-^{*p}$ map.

The core of the issue is that we need to guarantee that the "fresh" part of the $-^{*p}$ map consists only of elements which we can cancel off using the inherited $-^{*p}$ map from $K(n)$ on $\mathfrak{m}^{r-1}/\mathfrak{m}^r$. We have not established this control. It seems that a map of spectral sequences may help force this to be the case, but this has not yet been reasoned out.

Assuming that this is possible, the elements $f_I$ for $I$ not of the above form can be corrected to cycles by using $\gamma_I$ with $\gamma_I^{[1]}$ restricting to $f_I^{[1]}$ in the case $r = 1$. Transporting these classes to the bar construction is not hard; the same analysis of the relators can be applied there. Tate's construction was primarily useful in organizing the higher order homology classes. We have thus computed $H_{*,*}(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q-1)$.

Now we need to understand the differentials in this spectral sequence. Recall our pairing

$$(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q-2) \otimes (E/\mathfrak{m}^r)_* B\mathbb{Z}/p \xrightarrow{\circ} (E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q-1).$$

In the spectral sequence, this induces a map

$$H_{*,*}(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q-2) \otimes (E/\mathfrak{m}^r)_* B\mathbb{Z}/p \xrightarrow{\circ} H_{*,*}(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q-1)$$

compatible with the differentials in that $d^s(x \circ y) = d^s(x) \circ y$. This means that the differentials present in $H_{*,*}(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q-2)$ help determine the differentials in $H_{*,*}(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p, q-1)$, which we can use once we understand the action of the $\circ$-product on the bar construction, i.e., on the $E^1$ page.

**Theorem 6:** Let $0 \leq I_1 < \cdots < I_{q-1} < n$ be a multi-index and set $\tilde{I} = (I_2, \ldots, I_{q-1})$. Then we calculate

$$\gamma_{\tilde{I}} \circ a_{(I_1+1)} = (-1)^{q-2} \gamma_I.$$

*Non-proof.* Ravenel and Wilson use a degree argument, which we believe can be extended to the case of $E/\mathfrak{m}^2$; extending to arbitrary $r > 1$ seems much less feasible. They also note that they don't think theirs is the "right" way to handle this lemma, but we've had no better luck so far. $\square$

But, once this has been handled, the following results should be straightforward:

**Theorem 7:** For $I$ as above, we have $\gamma_{\tilde{I}}^{[p^i]} \circ a_{(I_1+i+1)} = (-1)^{q-2} \gamma_I^{[p^i]}$, modulo decomposables.

**Theorem 8:** Let $0 \leq I_1 < \cdots < I_q < n$, and let $\tilde{I} = (I_2, \ldots, I_q)$. Then $(-1)^{q-1} \gamma_{\tilde{I}-I_1-1}^{[p^{I_1}]}$ supports no nonvanishing differentials. Moreover, this class represents $a_I$ in the $E^\infty$ page.

**Theorem 9:** Select a multi-index satisfying $0 \leq I_1 < \cdots < I_{q-1} < n-1$, and set $K = (I_2, I_3, \ldots, I_{q-1}, n-1)$. Then

$$d^{2p^{I_1+1}-1} \gamma_{K-I_1-1}^{[p^{I_1+1}]}.$$

*Proofs.* These proofs are straightforward generalizations of several theorems of Ravenel and Wilson [8, Lemma 9.6-7]. $\square$

We have now completely described the structure of the bar spectral sequence, so we can read off what it says about $(E/\mathfrak{m}^2)_* K(\mathbb{Z}/p, q)$. For $0 \leq I_1 < \cdots < I_q < n$, the element $a_I$ certainly does appear in $(E/\mathfrak{m}^2)_* K(\mathbb{Z}/p, q)$. The Verschiebung can be described as $V(a_I) = a_{I-1}$ using $\circ$-equivariance, and its interaction with the Frobenius describes the map $-^{*p}$ by lifting the Frobenius from $(E/\mathfrak{m}^r)_* B\mathbb{Z}/p$. Finally, the coproduct description follows from Hopf ring properties. Each $a_I$ is represented in $E_{*,*}^\infty$ as explained in the result above, and this computation of $-^{*p}$ solves the multiplicative extension problem.

## 4. Future work

In addition to the gaps above, we list some questions left open by our investigation.

**4.1. The second inductive step.** Once the details above have been settled, the case $j > 1$ should follow quickly, and this will complete the description of the tower of algebras $(E/\mathfrak{m}^r)_* K(\mathbb{Z}/p^j, q)$. Computing the inverse limit will accomplish our original goal.

**4.2. Arbitrary square-zero deformations of derived local rings.** Once we establish what kind of control we need to complete the original argument, one generalization could be to work with an arbitrary derived square-zero extension rather than $E/\mathfrak{m}^r \to E/\mathfrak{m}^{r-1}$. Here are some conjectural theorem statements:

**Theorem 10:** Let $R$ be a complex-oriented ring spectrum so that $\pi_{-*}R$ is an $\mathbb{F}_p$-algebra and $\mathbb{C}P_R^\infty$ is $p$-divisible. Then $(B\mathbb{Z}/p^j)^R$ is described by the kernel of the map $- \frown [p^j]_R(x) : R_*\mathbb{C}P^\infty \to R_*\mathbb{C}P^\infty$, where $x$ is the element of the complex orientation. (In fact, this ought to admit a coordinate-free description.)

**Theorem 11:** Let $R$, $R_0$, and $k$ be $E_\infty$-rings so that $\pi_*R$ and $\pi_*R_0$ are complete, local, augmented $\pi_*k$-algebras. Let $R \to R_0$ be a map of complex oriented, $E_\infty$-ring spectra inducing a surjective, continuous, formally smooth map $\pi_*R \twoheadrightarrow \pi_*R_0$ of rings, whose kernel a square-zero ideal $M \subset \pi_*R$. If $K(\mathbb{Z}/p^j, q)^{R_0}$ is described by R-W, then so is $K(\mathbb{Z}/p^j, q)^R$.

**4.3. Connections and parallel transport.** It would be nice to think of $\Lambda^q\mathbb{C}P_{E_n}^\infty[p^\infty]$ as being like a bundle over the Lubin-Tate space. Ravenel and Wilson computed the fiber of this bundle at the geometric point corresponding to $K(n)$, and what we are now trying is to use deformations to compute the rest of this bundle; this is going to be very much like recovering a vector bundle from a connection on a manifold without monodromy. It would also be nice to flesh out exactly what "bundle" means here, how the map $E/\mathfrak{m}^r \to E/\mathfrak{m}^{r-1}$ acts like bundle restriction, and how our computation is limiting the available connections on the base scheme.

**4.4. Spectral sequences from algebraic geometry.** It may be possible to produce the bar spectral sequence on the algebraic side, then to use a map of spectral sequences to control the topological bar spectral sequence.

**4.5. Dieudonné crystals and $(E_n)_*K(\mathbb{Z}, *)$.** In the language of Dieudonné crystals, the move from $\mathbb{F}_p$-algebras to $\mathbb{Z}_{(p)}$-algebras has to do with the presence of Hodge structure; phrasing our results in the language of crystals may provide insight on what to do here. See [1, 5].

**4.6. Cohomology theories associated to $p$-divisible groups.** Once the case of Morava $E$-theory has been sorted, Lurie has recently produced a method for constructing cohomology theories associated to $p$-divisible groups with nontrivial étale part; this is discussed intelligibly by Goerss [4], but the real reference is somewhere in Lurie [7]. Translating our results to his derived setting is an ultimate, distant goal.

## References

[1] Matthew Ando, *Dieudonné crystals associated to Lubin-Tate formal groups*. Unpublished, 1994.

[2] Matthew Ando and Neil Strickland, *Weil Pairings and Morava K-theory*. Topology, Volume 40, Number 1, January 2001, pp. 127-156.

[3] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*. Volume 47 of Mathematical Surveys and Monographs, American Mathematical Society, 1997.

[4] Paul Goerss, *Topological modular forms, after Hopkins, Miller, and Lurie*. Séminaire BOURBAKI, 61ème année, no. 1005, 2008-2009.

[5] Alexander Grothendieck, *Groups de Barsotti-Tate et cristaux de Dieudonné*.

[6] Jonathan Lubin and John Tate, *Formal moduli for one-parameter formal Lie groups*. Bulletin de la Société Mathématique de France, v. 94, pp. 49-60, 1966.

[7] Jacob Lurie, *A Survey of Elliptic Cohomology*. Unpublished, 2007.

[8] Douglas Ravenel and W. Stephen Wilson, *The Morava K-Theories of Eilenberg-Maclane Spaces and the Conner-Floyd Conjecture*. American Journal of Mathematics, Vol. 102, No. 4 (Aug., 1980), pp. 691-74.

[9] John Tate, *Homology of Noetherian Rings and Local Rings*. Illinois Journal of Mathematics, 1957.