

p -adic congruences in iterated derivatives of \wp

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Abstract

We construct a p -adic measure valued in integral q -expansions of meromorphic Jacobi forms in an exponential coordinate, extending Katz's Eisenstein measure [Kat77] and the Mazur measure. We give an explicit formula for its moments in terms of the Weierstrass elliptic function.

We produce this measure via homotopy theory. The machinery of [AHRb] was developed with the intention of taking congruence relations as input and producing \mathbb{E}_∞ -orientations as output. We run their machine in reverse, using as input the recent results of [CL25] on the existence of \mathbb{E}_∞ -orientations of Tate fixed-point objects.

$q = 0$	t^0	t^{-1}	t^{-2}	t^{-3}	t^{-4}	t^{-5}
$M_1^{\text{Wit}^\sharp}$...000000 ₂	...111110 ₂	...111110 ₂	...111100 ₂	...111000 ₂	...110000 ₂
$M_2^{\text{Wit}^\sharp}$...101010 ₂	...111000 ₂	...111000 ₂	...100000 ₂	...110000 ₂	...000000 ₂
$M_3^{\text{Wit}^\sharp}$...000000 ₂	...100110 ₂	...100110 ₂	...111100 ₂	...101000 ₂	...100000 ₂
$M_4^{\text{Wit}^\sharp}$...011010 ₂	...110000 ₂	...110000 ₂	...000000 ₂	...100000 ₂	...000000 ₂
$M_5^{\text{Wit}^\sharp}$...000000 ₂	...001110 ₂	...001110 ₂	...111100 ₂	...011000 ₂	...010000 ₂
$M_6^{\text{Wit}^\sharp}$...001010 ₂	...101000 ₂	...101000 ₂	...100000 ₂	...010000 ₂	...000000 ₂
$M_7^{\text{Wit}^\sharp}$...000000 ₂	...110110 ₂	...110110 ₂	...111100 ₂	...001000 ₂	...000000 ₂
$M_8^{\text{Wit}^\sharp}$...111010 ₂	...100000 ₂	...100000 ₂	...000000 ₂	...000000 ₂	...000000 ₂
$M_9^{\text{Wit}^\sharp}$...000000 ₂	...011110 ₂	...011110 ₂	...111100 ₂	...111000 ₂	...110000 ₂
$M_{10}^{\text{Wit}^\sharp}$...101010 ₂	...011000 ₂	...011000 ₂	...100000 ₂	...110000 ₂	...000000 ₂
$M_{11}^{\text{Wit}^\sharp}$...000000 ₂	...000110 ₂	...000110 ₂	...111100 ₂	...101000 ₂	...100000 ₂
$M_{12}^{\text{Wit}^\sharp}$...011010 ₂	...010000 ₂	...010000 ₂	...000000 ₂	...100000 ₂	...000000 ₂
$M_{13}^{\text{Wit}^\sharp}$...000000 ₂	...101110 ₂	...101110 ₂	...111100 ₂	...011000 ₂	...010000 ₂
$M_{14}^{\text{Wit}^\sharp}$...001010 ₂	...001000 ₂	...001000 ₂	...100000 ₂	...010000 ₂	...000000 ₂
$M_{15}^{\text{Wit}^\sharp}$...000000 ₂	...010110 ₂	...010110 ₂	...111100 ₂	...001000 ₂	...000000 ₂
$M_{16}^{\text{Wit}^\sharp}$...111010 ₂	...000000 ₂				
$M_{17}^{\text{Wit}^\sharp}$...000000 ₂	...111110 ₂	...111110 ₂	...111100 ₂	...111000 ₂	...110000 ₂
$M_{18}^{\text{Wit}^\sharp}$...101010 ₂	...111000 ₂	...111000 ₂	...100000 ₂	...110000 ₂	...000000 ₂
$M_{19}^{\text{Wit}^\sharp}$...000000 ₂	...100110 ₂	...100110 ₂	...111100 ₂	...101000 ₂	...100000 ₂
$M_{20}^{\text{Wit}^\sharp}$...011010 ₂	...110000 ₂	...110000 ₂	...000000 ₂	...100000 ₂	...000000 ₂
$M_{21}^{\text{Wit}^\sharp}$...000000 ₂	...001110 ₂	...001110 ₂	...111100 ₂	...011000 ₂	...010000 ₂
$M_{22}^{\text{Wit}^\sharp}$...001010 ₂	...101000 ₂	...101000 ₂	...100000 ₂	...010000 ₂	...000000 ₂
$M_{23}^{\text{Wit}^\sharp}$...000000 ₂	...110110 ₂	...110110 ₂	...111100 ₂	...001000 ₂	...000000 ₂
$M_{24}^{\text{Wit}^\sharp}$...111010 ₂	...100000 ₂	...100000 ₂	...000000 ₂	...000000 ₂	...000000 ₂
$M_{25}^{\text{Wit}^\sharp}$...000000 ₂	...011110 ₂	...011110 ₂	...111100 ₂	...111000 ₂	...110000 ₂
$M_{26}^{\text{Wit}^\sharp}$...101010 ₂	...011000 ₂	...011000 ₂	...100000 ₂	...110000 ₂	...000000 ₂
$M_{27}^{\text{Wit}^\sharp}$...000000 ₂	...000110 ₂	...000110 ₂	...111100 ₂	...101000 ₂	...100000 ₂
$M_{28}^{\text{Wit}^\sharp}$...011010 ₂	...010000 ₂	...010000 ₂	...000000 ₂	...100000 ₂	...000000 ₂
$M_{29}^{\text{Wit}^\sharp}$...000000 ₂	...101110 ₂	...101110 ₂	...111100 ₂	...011000 ₂	...010000 ₂
$M_{30}^{\text{Wit}^\sharp}$...001010 ₂	...001000 ₂	...001000 ₂	...100000 ₂	...010000 ₂	...000000 ₂
$M_{31}^{\text{Wit}^\sharp}$...000000 ₂	...010110 ₂	...010110 ₂	...111100 ₂	...001000 ₂	...000000 ₂

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1 Introduction

Definition 1.1 ([Kat77, Sections 3.1–2]). For A a torsion-free \mathbb{Z}_p -algebra, an A -valued p -adic moment sequence is a sequence $M_n \in A$ such that for any rational polynomial $f(r) = \sum_n a_n r^n \in \mathbb{Q}[r]$ with $f(\mathbb{Z}_p^\times) \subseteq \mathbb{Z}_p$, the sum $\sum_n a_n M_n \in A \otimes \mathbb{Q}$ is an A -integer.

Moment sequences arise out of p -adic integrals via the correspondence

$$M_n := \int_{\mathbb{Z}_p^\times} dM r^n, \quad \int_{\mathbb{Z}_p^\times} dM \sum_{n=0}^N a_n r^n := \sum_{n=0}^N a_n M_n.$$

A -integrality can be interpreted as a family of *Kummer congruences* on the sequence elements. For example, the polynomial $p^{-1}(r^p - r)$ is p -adically integer-valued, hence $p^{-1}(M_p - M_1)$ must be an A -integer, or equivalently $M_p \equiv M_1 \pmod{p}$. Other choices of test polynomial yield other congruences.

We prove the following:

Main Theorem A. *Pick any $c \in \mathbb{Z}_p^\times \setminus \{\pm 1\}$. There is a $\mathbb{Z}[[q]]((t))_p^\wedge$ -valued moment sequence whose terms for $n \geq 1$ are given by*

$$\begin{aligned} M_1^{\text{Wit}^\sharp} - M_1^{\text{Wit}} &= \mu_1(q, t) + (1 - c) \cdot \frac{\log(1 - t)}{(2\pi i)^2} \cdot (G_2(q) - p \cdot G_2(q^p)), \\ M_2^{\text{Wit}^\sharp} - M_2^{\text{Wit}} &= \mu_2(q, t) + (1 - c^2) \cdot \frac{-1}{2(2\pi i)^2} \cdot (G_2(q) - p \cdot G_2(q^p)), \\ M_{n \geq 3}^{\text{Wit}^\sharp} - M_{n \geq 3}^{\text{Wit}} &= \mu_n(q, t), \end{aligned}$$

where

$$\mu_n(q, t) = \frac{1 - c^n}{(2\pi i)^n} \cdot \left(\wp_q^{(n-2)} \left(\frac{\log(1 - t)}{-2\pi i} \right) - \wp_{q^p}^{(n-2)} \left(\frac{p \log(1 - t)}{-2\pi i} \right) \right),$$

\wp_q is the q -expansion of the Weierstrass elliptic function, $\wp_q^{(n)}$ is its n^{th} derivative, and G_2 is the second Eisenstein series.

As warm-up, we also prove the following theorem:

Main Theorem B. *Pick any $c \in \mathbb{Z}_p^\times \setminus \{\pm 1\}$. There is a $\mathbb{Z}((t))_p^\wedge$ -valued moment sequence whose terms for $n \geq 1$ are given by*

$$M_n^{\text{Td}\#} - M_n^{\text{Td}} = (1 - c^n) \sum_{m=1}^n \frac{\Delta^m[r^n](0)}{m} \left(t^{-m} - \frac{p^{n-1}t^{-pm}}{((-1)^{p+1} - p\zeta)^m} \right),$$

where $\Delta^m[r^n](0)$ is the m^{th} forward finite difference of the test polynomial r^n at zero (i.e., its m^{th} Mahler coefficient), and where $\zeta \in \mathbb{Z}[t^{-1}]$ is defined by $(t^{-1} - 1)^p = t^{-p} + (-1)^p + p\zeta$.

Remark 1.2. The p in front of ζ in $((-1)^{p+1} - p\zeta)^{-m}$ yields a convergent sum in $\mathbb{Z}((t))_p^\wedge$, despite the unboundedly increasing powers of t^{-1} .

Remark 1.3. Main Theorem B is essentially the component of Main Theorem A induced by the pole of $\wp^{(n-2)}$. More precisely:

- The terms of $M_n^{\text{Wit}\#}$ and $M_n^{\text{Td}\#}$ of nonpositive t -degree match exactly.

We can also identify other subsectors of terms:

- The terms of $M_n^{\text{Wit}\#}$, $M_n^{\text{Td}\#}$, M_n^{Wit} , and M_n^{Td} of vanishing t - and q -degree are the moments of the Mazur measure, which p -adically interpolates the Bernoulli numbers [Lan90, Section 4.3].
- The terms of $M_n^{\text{Wit}\#}$ and M_n^{Wit} of vanishing t -degree are those of Katz's Eisenstein measure [Kat77, Lemma 3.5.6], appearing also in Ando, Hopkins, and Rezk [AHRb, Section 10].
- The sectors of terms not mentioned above or in the Main Theorems vanish.

Remark 1.4. The Jacobi orientation has captured the attention of many mathematicians (e.g., [Hir88, Zag88, Wit94, KYY94, Gri99, BL00, BL03, AFG08, Lib11, BCK⁺16, LY25]). Until recently [CL25], it had been an open question whether it refines to a map of \mathbb{E}_∞ -ring spectra, which would make it accessible to the techniques of higher algebra and would induce relations among its special values (e.g., [AHS04]). Ando, Hopkins, and Rezk [AHRb] provide the machine traditionally used to resolve such questions, which for the Jacobi orientation takes as input precisely Main Theorem A—so, these congruences have been of interest to topologists, geometers, and physicists for some time. (Our proof, however, *concludes* the congruences *from* the separately known \mathbb{E}_∞ -orientation.)

Remark 1.5. See Figure 1 and Figure 2 for visualizations of these congruences at $p = 3$, and Figure 3 and Figure 4 (which extends the title figure) for $p = 2$.

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2 Résumés

Our main theorems are largely an application of existing work, especially a recent paper of the first author [CL25]. Accordingly, we recall below the inputs which we require to complete our calculation.

2.1 Résumé on elliptic functions

It will be convenient to have formulas on hand for a variety of standard functions appearing in the theory of complex elliptic curves. To fix terms, let us consider (twisted) functions on the compactified moduli of elliptic curves, referred to as *modular forms*. Such functions are typically presented in three ways [Sil94, Section I.1, Remark 3.3]:

1. A modular form f of weight k is a function on choices of lattice $\Lambda \subset \mathbb{C}$ with the property $f(c \cdot \Lambda) = c^{-k} f(\Lambda)$ and which extends to a meromorphic function at $\text{span}\{+i \cdot \infty, 1\}$.
2. Selecting a basis $\{\tau, 1\}$ for the (possibly positively rescaled) lattice Λ , f can be understood as a function of $\tau \in \mathfrak{h}$ satisfying $f(M \cdot \tau) = (c\tau + d)^{-k} f(\tau)$ and extending to a meromorphic function at $\tau = +i \cdot \infty$.
3. Using periodicity against $\text{SL}_2\mathbb{Z}$ and meromorphicity at $\tau = +i \cdot \infty$, the q -expansion of f is its Fourier series, written in terms of $q = e^{2\pi i\tau}$ as a bounded-below sum $\sum_{n=-m}^{\infty} a_n q^n$.¹

We will also be interested in functions on a versal elliptic curve, called *Jacobi forms* [EZ85, Section I.1], modeled as functions on $\mathbb{C} \times \mathfrak{h}$ equivariant for $\text{SL}_2\mathbb{Z} \ltimes \mathbb{Z}^2$. Such functions again can be parameterized in terms of Λ , τ , or q .

Example 2.1. For $k \geq 2$, the k^{th} *Eisenstein series* G_k is the (quasi²)modular form defined by the lattice sum [Kat77, Equation 2.2.5]

$$G_k(\Lambda) = \sum_{\lambda \in \Lambda^*} \frac{1}{\lambda^k}.$$

It is evidently modular of weight k . Note that if k is odd, the sum telescopes to zero. In the even case, the q -expansion of G_{2k} is

$$\frac{(2k-1)!}{(2\pi i)^{2k}} \cdot G_{2k}(q) = -\frac{B_{2k}}{2k} + 2 \cdot \sum_{j=1}^{\infty} \sigma_{2k-1}(j) q^j,$$

where B_{2k} is the $2k^{\text{th}}$ Bernoulli number, defined by $\sum_{n=0}^{\infty} B_n x^n = \frac{x}{e^x - 1}$, and where $\sigma_{2k-1}(j) = \sum_{d|j} d^{2k-1}$ is the divisor-sum function [Kat77, Equation 2.4.8].

¹It is not trivial to discern when an arbitrary q -series is the q -expansion of a modular form.

² $G_2(\tau)$ does not quite transform as a modular form, but instead acquires an extra summand.

Example 2.2. The Weierstrass σ -function is defined by [Sil94, Proposition 5.4.a]

$$\sigma_\Lambda(z) = z \cdot \prod_{\lambda \in \Lambda^*} \left[\left(1 - \frac{z}{\lambda}\right) \cdot \exp \left(\frac{z}{\lambda} + \frac{1}{2} \left(\frac{z}{\lambda} \right)^2 \right) \right].$$

Among other properties, it is designed so that its logarithm becomes a generating function for Eisenstein series [Kat77, Equation 2.2.7]:

$$\log z - \log \sigma_\Lambda(z) = \sum_{n=3}^{\infty} \frac{(2\pi iz)^n}{n!} \cdot \left(\frac{(n-1)!}{(2\pi i)^n} \cdot G_n(\Lambda) \right).$$

We will need a product formula for its q -expansion [Sil94, Theorems 6.4, 6.3.b]:

$$2\pi i \cdot \sigma_q(z) = \exp \left(\frac{1}{2} (2\pi i \cdot z + G_2(q) \cdot z^2) \right) \cdot (1-u) \cdot \prod_{j=1}^{\infty} \frac{(1-q^j u)(1-q^j u^{-1})}{(1-q^j)^2},$$

where $u = e^{-2\pi iz}$. Lastly, the second logarithmic derivative of σ is named the Weierstrass \wp -function [Sil94, Proposition 5.4.c]:

$$\wp_\Lambda(z) = -\frac{d^2}{dz^2} \log \sigma_\Lambda(z),$$

which is a meromorphic Jacobi form of index 0 and weight 2.

2.2 Résumé on elliptic cohomology

In this subsection we recall some basic elements of complex orientation theory.

Definition 2.3. Let R be a homotopy ring spectrum. The following two pieces of data are equivalent:

1. A *complex orientation* of R is a homotopy ring map $\mathrm{MU} \rightarrow R$ from the complex bordism spectrum [Ada74, Lemma I.4.6].
2. A *coordinate* is a class $\xi \in R^2 \mathbb{CP}^\infty$ which restricts to $\Sigma^2 1 \in R^2 S^2$ along $S^2 \simeq \mathbb{CP}^1 \rightarrow \mathbb{CP}^\infty$ [Ada74, I.2.1].

A coordinate induces an isomorphism $R_*[\xi] \rightarrow R^* \mathbb{CP}^\infty$. The Künneth formula then gives an isomorphism $R^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq R_*[\xi_l, \xi_r]$ where ξ_l and ξ_r are the pullbacks of ξ along the two projections. Let $\mu: \mathbb{CP}^\infty \times \mathbb{CP}^\infty \rightarrow \mathbb{CP}^\infty$ be the multiplication map classifying the tensor product of line bundles. The *formal group law* $F(\xi_l, \xi_r)$ is defined to be the bivariate power series associated to the element $\mu^* \xi$ [Ada74, Lemma I.2.7]. We often denote it by $\xi_l +_F \xi_r$.

Example 2.4. Note that MU is a connective ring spectrum with $\pi_0 \mathrm{MU} = \mathbb{Z}$, so the Postnikov truncation functor induces a ring map $\mathrm{MU} \rightarrow \mathbb{Z}$. Thus, if R is a \mathbb{Z} -algebra, there is a canonical orientation $\mathrm{MU} \rightarrow \mathbb{Z} \rightarrow R$. We will refer to this as the *additive* orientation of R , as its group law is $F(x_l, x_r) = x_l + x_r$.

Example 2.5. Write $\pi_* \mathrm{KU} = \mathbb{Z}[\beta^\pm]$, $\pi_* \mathrm{KU}_{\mathrm{Tate}} = \mathbb{Z}[\beta^\pm][q]$ with $|\beta| = 2$, $|q| = 0$.

- The *Todd orientation* ω_{Td} : $\mathrm{MU} \rightarrow \mathrm{KU}$ has coordinate [Ada74, I.2.3]

$$\xi_{\mathrm{Td}} := \beta^{-1}(1 - [L]) \in \mathrm{KU}^2 \mathbb{CP}^\infty.$$

- The *Witten orientation* ω_{Wit} : $\mathrm{MU} \rightarrow \mathrm{KU}_{\mathrm{Tate}}$ has coordinate [Zag88, (6)]

$$\xi_{\mathrm{Wit}} = \beta^{-1}(1 - [L]) \prod_{j=1}^{\infty} \frac{(1 - [L]q^j)(1 - [L^{-1}]q^j)}{(1 - q^j)^2} \in \mathrm{KU}_{\mathrm{Tate}}^2 \mathbb{CP}^\infty.$$

For later convenience we also define the degree zero element $s := \beta \xi_{\mathrm{Td}}$.

Remark 2.6 ([AHS01, Section 2.7]). An *elliptic cohomology theory* is a complex-oriented theory with an isomorphism between its formal group law and that of some (possibly singular) elliptic curve. $\mathrm{KU}_{\mathrm{Tate}}$ can be understood as an elliptic theory, with formal group law inherited from the Tate curve, as can KU via the nodal cubic at the center $q = 0$.

Definition 2.7 ([Ada74, Corollary I.7.15]). Let $\omega: \mathrm{MU} \rightarrow R$ be an orientation, which begets an orientation $\omega_{\mathbb{Q}}: \mathrm{MU} \rightarrow R \otimes \mathbb{Q}$. Since $R \otimes \mathbb{Q}$ is a \mathbb{Z} -algebra, Example 2.4 endows it with an additive coordinate x . Writing ξ_ω for the coordinate so associated to $\omega_{\mathbb{Q}}$, it can be expressed in terms of x as a power series of the form $\exp_\omega(x) := \xi_\omega(x) = x + O(x^2)$. This series is called the *exponential* of the formal group law F_ω associated to ω because it satisfies

$$\exp_\omega(x_l) +_{F_\omega} \exp_\omega(x_r) = \exp_\omega(x_l + x_r).$$

Example 2.8 ([Hir66, Chapter 3]). The exponential of the Todd orientation is

$$\exp_{\mathrm{Td}}(x) = \beta^{-1}(1 - e^{-\beta x}).$$

Therefore we have

$$\exp_{\mathrm{Wit}}(x) = \beta^{-1}(1 - e^{-\beta x}) \prod_{j=1}^{\infty} \frac{(1 - e^{-\beta x}q^j)(1 - e^{\beta x}q^j)}{(1 - q^j)^2}.$$

Definition 2.9 ([AHRb, Section 3.4]). The *nepers* of an orientation $\omega: \mathrm{MU} \rightarrow R$ are the elements $N_n^\omega \in \pi_{2n} R \otimes \mathbb{Q}$ defined by the formula

$$\log \left(\frac{x}{\exp_\omega(x)} \right) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \cdot N_n^\omega.$$

When R has a periodicity element $\beta \in \pi_2 R$, meaning $\pi_* R \cong (\pi_0 R)[\beta^\pm]$, it will often be convenient to work with the degree zero element $\beta^{-n} N_n^\omega$ instead of N_n^ω .

Remark 2.10 ([AHRb, Proposition 3.12]). The division of the two Thom classes for the universal line bundle $L \rightarrow \mathbb{CP}^\infty$ is referred to as the *Hirzebruch series* of ω . One can form the same division of Thom classes for any vector bundle, and the neper N_n^ω is so obtained by the bundle over S^{2n} corresponding to a generator of $\pi_{2n} \mathrm{BU}$.

Next, we outline a construction for getting new orientations from old ones, due to Ando, French, and Ganter [AFG08]. First, we recall a fundamental piece of homotopy theory.

Definition 2.11 (e.g., [GM95], [NS18]). Let R be a complex oriented ring spectrum with coordinate γ . The *Tate fixed-point spectrum* $R^{t\mathbb{T}}$ is the ring spectrum obtained by inverting the element γ in the ring spectrum $R^{\mathbb{CP}^\infty}$. In particular $\pi_* R^{t\mathbb{T}}$ is the Laurent series ring $R_*((\gamma))$, and there is a *unit* map $u: R \rightarrow R^{t\mathbb{T}}$ of ring spectra, which induces the obvious inclusion $R_* \rightarrow R_*((\gamma))$ on homotopy groups.

Definition 2.12 ([AFG08]). Let $\omega: \mathrm{MU} \rightarrow R$ be a complex orientation with its induced isomorphism $\pi_* R^{t\mathbb{T}} \simeq R_*((\gamma))$. Define the *unit orientation* of $R^{t\mathbb{T}}$ as the composite

$$\mathrm{MU} \xrightarrow{\omega} R \xrightarrow{u} R^{t\mathbb{T}},$$

and let ξ and $+_F$ denote the corresponding coordinate and formal group law. Define the *sharped orientation* $\omega^\sharp: \mathrm{MU} \rightarrow R^{t\mathbb{T}}$ to be the ring map associated to the coordinate

$$\xi \cdot \frac{\gamma}{\xi +_F \gamma}.$$

Let y be such that $\exp_\omega(y) = \gamma$. Then the sharped exponential satisfies

$$\exp_{\omega^\sharp}(x) = \exp_\omega(x) \cdot \frac{\exp_\omega(y)}{\exp_\omega(x+y)}.$$

Remark 2.13 ([AFG08, Sections 6–7]). The *Jacobi orientation* arises as the sharp construction applied to the Witten orientation. On homotopy, it takes values in meromorphic Jacobi forms of index 0.

2.3 Résumé on \mathbb{E}_∞ –rings

Whereas in classical algebra commutativity is a *property* of a ring and maps of rings are automatically compatible with any commutativity present, a ring spectrum in homotopy theory requires *extra data* to witness its commutativity, as do maps of commutative ring spectra. A ring spectrum with such extra structure is called an \mathbb{E}_∞ –ring, and such a map an \mathbb{E}_∞ –ring map. The complex bordism spectrum MU is naturally an \mathbb{E}_∞ –ring ([Lew78], [May77, Chapter III]), and thus if R is an \mathbb{E}_∞ –ring one can ask when a complex orientation $\mathrm{MU} \rightarrow R$ admits an \mathbb{E}_∞ –structure. This question has received much interest and has seen recent progress ([AHRb], [HL18], [HY20], [Bal23], [BSY24], ...), but it is usually quite difficult—the number of known \mathbb{E}_∞ –orientations is small, the crown jewel of which is the *string orientation* of tmf.

Our interest comes from a result of the first author:

Theorem 2.14 ([CL25, Theorem 5.14]). *The sharp construction of Definition 2.12 sends \mathbb{E}_∞ –orientations to \mathbb{E}_∞ –orientations.* \square

Therefore the sharped string orientation (or any of the handful of \mathbb{E}_∞ -orientations it begets, cf. Theorem 2.20) automatically admits this extra \mathbb{E}_∞ -structure, and we would like to understand it in detail. Ultimately, we will conclude that it entails the congruences advertised in the introduction.

Let us describe others' pursuit of the program to express this \mathbb{E}_∞ -structure in a manageable form for palatable target rings. There is a distinguished class of spectra, called *K(1)-local*, whose \mathbb{E}_∞ -rings carry an extremely nice theory of cohomology operations, controlled by an essentially unique operation ψ^p [BMMS86, Chapter IX]. A key insight of Ando, Hopkins, and Rezk is that if R is *K(1)-local*, then an \mathbb{E}_∞ -structure on an orientation $\omega: \mathrm{MU} \rightarrow R$ is well captured by its interaction with ψ^p . Our rings of interest, $\mathrm{KU}^{t\mathbb{T}}$ and $\mathrm{KU}_{\mathrm{Tate}}^{t\mathbb{T}}$, become *K(1)-local* after p -completion. Their ψ^p maps are as follows:

Example 2.15 ([Ada62, Theorems 5.1.iii and 7.2]). Upon p -completion, the Todd and Witten orientations of Example 2.5 induce identifications

$$\pi_*(\mathrm{KU}^{t\mathbb{T}})_p^\wedge = \mathbb{Z}[\beta^\pm]((t))_p^\wedge, \quad \pi_*(\mathrm{KU}_{\mathrm{Tate}}^{t\mathbb{T}})_p^\wedge = \mathbb{Z}[\beta^\pm][[q]]((t))_p^\wedge,$$

where we have adjusted the Laurent element $t := \beta\gamma$ into degree 0 (cf. Definition 2.11). The operations ψ^p are the ring homomorphisms determined by

$$\psi^p(t) = 1 - (1 - t)^p, \quad \psi^p(q) = q^p, \quad \psi^p(\beta) = p\beta.$$

For a homotopy element $f_q(t) \cdot \beta^k \in \pi_{2k}\mathrm{KU}_{\mathrm{Tate}}^{t\mathbb{T}} \cong \mathbb{Z}[[q]]((t))_p^\wedge \{ \beta^k \}$ we then have

$$\psi^p(f_q(t) \cdot \beta^k) = p^k \cdot f_{q^p}(1 - (1 - t)^p) \cdot \beta^k.$$

We measure the interaction of ψ^p with the orientation ω like so:

Definition 2.16 ([AHRb, Proposition 7.10]). Pick any $c \in \mathbb{Z}_p^\times \setminus \{\pm 1\}$. The *moments* of an orientation $\omega: \mathrm{MU} \rightarrow R$ of a *K(1)-local* ring are the elements

$$M_n^\omega = (1 - c^n) \left(\mathrm{id} - \frac{\psi^p}{p} \right) (N_n^\omega) \in \pi_{2n}R \otimes \mathbb{Q}.$$

As with Definition 2.9, when R has a periodicity element $\beta \in \pi_2R$, it will often be convenient to work with the degree 0 element $\beta^{-n}M_n^\omega$. However, one has to take a bit of care, since $\psi^p(\beta)$ need not be equal to β (as Example 2.15 shows).

Remark 2.17 ([Rez06]). The operator $\mathrm{id} - p^{-1}\psi^p$ is part of a more general theory of *logarithmic cohomology operations*.

We can now state a key result of Ando–Hopkins–Rezk:

Theorem 2.18 ([AHRb, Theorem 6.1], see also [AHRa], [Pet19, Appendix A.4]). *Let $\omega: \mathrm{MU} \rightarrow R$ be an \mathbb{E}_∞ -orientation of a *K(1)-local* ring R which is flat over KU_p^\wedge . The moments $M_{n \geq 1}^\omega$ belong to an R_* -valued p -adic moment sequence. \square*

Remark 2.19 ([AHRb, Lemma 7.14, Proposition 14.6.2, Section 3.5], [Mil82]). Orientations are automatically suitably compatible with the unit map of the ring spectrum, which manifests in the moment sequence as the identity

$$\lim_{j \rightarrow \infty} M_{(p-1)p^j}^\omega = M_0^\omega = p^{-1} \log c^{p-1}.$$

Finally, we need some existing \mathbb{E}_∞ -orientations of KU and $\mathrm{KU}_{\mathrm{Tate}}$ to which we will apply Theorem 2.14 to get \mathbb{E}_∞ -orientations of $(\mathrm{KU}^{t\mathbb{T}})_p^\wedge$ and $(\mathrm{KU}_{\mathrm{Tate}}^{t\mathbb{T}})_p^\wedge$, and then further apply Theorem 2.18 to get p -adic moment sequences. We record the following theorem.

Theorem 2.20 ([AHRb, Theorem 10.3, Proposition 10.10]). *After p -completion, the orientations of Example 2.5*

$$\omega_{\mathrm{Td}}: \mathrm{MU} \rightarrow \mathrm{KU}_p^\wedge, \quad \omega_{\mathrm{Wit}}: \mathrm{MU} \rightarrow (\mathrm{KU}_{\mathrm{Tate}})_p^\wedge$$

admit \mathbb{E}_∞ -structures. □

Remark 2.21. To be precise, the statement of Proposition 10.10 of [AHRb] produces an \mathbb{E}_∞ -structure on the $\mathrm{MU}\langle 6 \rangle$ -orientation associated to ω_{Wit} . However, the same proof, with the target ring changed to $\pi_*\mathrm{KU}_{\mathrm{Tate}}$, shows that ω_{Wit} itself admits an \mathbb{E}_∞ -orientation. Alternatively, one can use obstruction theory along the Whitehead tower of bu to push the \mathbb{E}_∞ -structure from $\mathrm{MU}\langle 6 \rangle$ to MU .

3 Proofs of the Main Theorems

We now apply this machinery to the sharped orientations of (Tate) K -theory from Theorem 2.20 to produce p -adic moment sequences. It will be convenient to fix the following notation:

$$2\pi iz = x = -\log(1-s) = -\log(u), \quad 2\pi i\alpha = y = -\log(1-t) = -\log(v).$$

Remark 3.1. While s agrees with the element defined at the end of Example 2.5, x does not quite agree with the additive coordinate of previous sections. Instead, it is the degree 0 element gotten by using the periodicity element β . This slight abuse of notation is well worth it, as the formulas that follow would otherwise be infested with powers of β .

Throughout, let $c \in \mathbb{Z}_p^\times \setminus \{\pm 1\}$ be a choice of p -adic unit.

3.1 Todd and Tod^\sharp

Lemma 3.2 ([Kob84, Theorem 7.2], [AHRb, Proposition 10.2]). *There is a p -adic moment sequence which in positions $n \geq 1$ is given by*

$$(1 - c^n) \cdot (1 - p^{n-1}) \cdot \frac{-B_n}{n},$$

where B_n denotes the n^{th} Bernoulli number.

Proof. Using Example 2.8, the nepers associated to the Todd orientation are

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot N_n^{\mathrm{Td}} = \log\left(\frac{x}{s}\right) = \log\left(\frac{x}{1 - e^{-x}}\right).$$

The series expansion of the right-hand side follows from:

$$\frac{d}{dx} \log \frac{x}{1 - e^{-x}} = \frac{1}{x} - \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{x} - \frac{1}{x} \cdot \frac{x}{e^x - 1} = \sum_{n=1}^{\infty} -B_n \frac{x^{n-1}}{n!}.$$

We then calculate the moments (cf. Definition 2.16) and find that they are the claimed sequence, which is a p -adic moment sequence by Theorem 2.18. \square

Proof of Main Theorem B. Combining Example 2.8 and Definition 2.12, the nepers associated to the Todd $^\sharp$ orientation are

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot N_n^{\text{Td}^\sharp} = \log \left(\frac{x}{s} \right) + \log \left(\frac{1 - e^{-x}(1-t)}{t} \right).$$

The first summand is the subject of Lemma 3.2, so we focus on the second:

$$\log \left(\frac{1 - t^{-1}(1 - e^x)}{e^x} \right) = -x - \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{m=1}^n \frac{t^{-m}}{m} \sum_{j=0}^m (-1)^j \binom{m}{j} j^n.$$

The inner sum is exactly the iterated forward finite difference $\Delta^m[r^n](0)$. Apply Theorem 2.18 to produce the p -adic moment sequence:

$$M_n^{\text{Td}^\sharp} - M_n^{\text{Td}} = (1 - c^n) \left(\sum_{m=1}^n \frac{\Delta^m[r^n](0)}{m} \left(t^{-m} - \frac{p^{n-1}}{(1 - (1-t)^p)^m} \right) \right).$$

The binomial raised to the m^{th} power does not naively expand, as $(1-t)^p$ is not a topologically nilpotent element of $\pi_*(\text{KU}^{t\mathbb{T}})_p^\wedge$. However, by defining ζ such that $(t^{-1} - 1)^p = t^{-p} + (-1)^p + p\zeta$, we may rewrite this as

$$M_n^{\text{Td}^\sharp} - M_n^{\text{Td}} = (1 - c^n) \sum_{m=1}^n \frac{\Delta^m[r^n](0)}{m} \left(t^{-m} - \frac{p^{n-1} t^{-pm}}{((-1)^{p+1} - p\zeta)^m} \right),$$

which does define a convergent series in $\pi_*(\text{KU}^{t\mathbb{T}})_p^\wedge$. \square

Remark 3.3. When $m > n$, the t^{-m} term in $N_n^{\text{Td}^\sharp}$ vanishes for two reasons: the m -fold forward finite difference operator annihilates r^n , and the term $(1 - e^{-x})^m$ appearing in the expansion of $\log(1 - t^{-1}(1 - e^x))$ has x -degree at least m .

3.2 Witten and Witten $^\sharp$

Lemma 3.4 ([Kat77, Lemma 3.5.6], [AHRb, Propositions 10.9–10]). *There is a $\mathbb{Z}[[q]]_p^\wedge$ -valued moment sequence which in even positions $2n \geq 1$ is given by*

$$\frac{(1 - c^{2n}) \cdot (2n - 1)!}{(2\pi i)^{2n}} \cdot (G_{2n}(q) - p^{2n-1} G_{2n}(q^p))$$

and which vanishes in odd positions $2n + 1 \geq 1$.

Proof. The exponential for the Witten orientation of Tate K -theory in Example 2.8 shows the associated sequence of nepers to be

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} \cdot N_n^{\text{Wit}} = \log\left(\frac{x}{1-u}\right) + \log\left(\prod_{j=1}^{\infty} \frac{(1-q^j)^2}{(1-uq^j)(1-u^{-1}q^j)}\right).$$

The first summand is felled by Lemma 3.2, so we concentrate on the second:

$$\begin{aligned} \log\left(\prod_{j=1}^{\infty} \frac{(1-q^j)^2}{(1-uq^j)(1-u^{-1}q^j)}\right) &= \sum_{j=1}^{\infty} \sum_{d=1}^{\infty} \frac{1}{d} (-2q^{jd} + q^{jd}e^{-dx} + q^{jd}e^{dx}) \\ &= 2 \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} \sum_{j=1}^{\infty} q^j \sigma_{2n-1}(j). \end{aligned}$$

Comparing with Example 2.1 shows

$$N_n^{\text{Wit}} = \frac{(2n-1)!}{(2\pi i)^{2n}} \cdot G_{2n}(q).$$

All that remains is to calculate the moments M_n^{Wit} and apply Theorem 2.18. \square

Proof of Main Theorem A. Again, combining Example 2.8 and Definition 2.12, the sharped Witten nepers are given by

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot N_n^{\text{Wit}^\sharp} = \log\left(\frac{x}{\exp_{\text{Wit}}(x)}\right) + \log\left(\frac{\exp_{\text{Wit}}(x+y)}{\exp_{\text{Wit}}(y)}\right).$$

The first summand is felled by Lemma 3.4, so we focus on expanding the second. Using Example 2.2, the Witten exponential can be written in terms of σ_q as

$$\exp_{\text{Wit}}(x) = 2\pi i \cdot \sigma_q\left(\frac{x}{2\pi i}\right) \cdot \exp\left(-\frac{1}{2}x - \frac{G_2(q)}{2 \cdot (2\pi i)^2}x^2\right),$$

and hence that second summand can be written as

$$\begin{aligned} \log\left(\frac{\exp_{\text{Wit}}(x+y)}{\exp_{\text{Wit}}(y)}\right) &= \left(-\frac{1}{2} + \frac{G_2(q)}{(2\pi i)^2} \log(1-t)\right) \cdot x + \frac{-G_2(q)}{2(2\pi i)^2} \cdot x^2 \\ &\quad + \sum_{n=1}^{\infty} \frac{x^n}{n!} \cdot \frac{1}{(2\pi i)^n} \cdot \wp_q^{(n-2)}\left(\frac{\log(1-t)}{-2\pi i}\right). \end{aligned}$$

The moments are then calculated to be exactly the expressions appearing in the Theorem statement, so Theorem 2.18 finishes the proof. \square

4 Further directions

Remark 4.1. The recipe followed here applies to any other \mathbb{E}_∞ -orientations the reader may have on hand.

- [AHRb]: The spectrum TMF is a topological incarnation of the moduli stack of elliptic curves over a general base. Its homotopy, analogous to the cohomology of the moduli stack, brings to light certain torsion phenomena not visible in the homotopy of $\pi_*\mathrm{KU}_{\mathrm{Tate}}$. Though TMF is no longer complex orientable, there is a σ -orientation which suitably refines the Witten orientation of $\mathrm{KU}_{\mathrm{Tate}}$ and is itself \mathbb{E}_∞ , suggesting one possible starting point for a generalization of our results.
- [HL16, Wil15, Sen23]: Topologists have also constructed variations on TMF for other congruence subgroups and other levels (though lifts of the σ -orientation have only been considered in limited situations). The number theory literature makes substantial use of these other stacks, and so an enrichment of our results to that setting would provide much better contact.
- [LY25]: Topologists have recently constructed a spectrum TJF which refines the theory of Jacobi forms analogously to how TMF refines the moduli of elliptic curves. This perhaps forms a more appropriate target for parametric Jacobi orientations than $\mathrm{TMF}^{t\mathbb{T}}$. (We thank Yamashita for bringing this to our attention.)
- [MT91, Appendix II]: Mazur and Tate construct functions analogous to the classical Weierstrass function, but over other adic rings and specified by quite different formulas. Since TMF lies over a general base, the σ -orientation suitably restricted may shed light on these other σ -functions.
- [Bal23], [BSY24]: Morava E -theories are known to carry \mathbb{E}_∞ -orientations, and the relevant logarithms bear a number-theoretic description [Rez06, 1.10]. To our knowledge, the associated congruences have not been explored—nevermind their sharped versions.

Remark 4.2. Number theorists have made strikingly productive use of p -adic interpolation in the context of special values of L -functions—to pluck some examples from a hat, see [Kat76, Kat77, Kat80, Yag82, MTT86, GS93], though this list is hardly exhaustive. At least to the authors’ untrained eyes, these results appear to require a greater parametricity (e.g.: in level, in ground ring) than is offered by our result on the standard Weierstrass elliptic functions. Still, we hope that it can perhaps be put to similar use in some more limited context, or that it inspires subsequent production of broader congruences.

Remark 4.3. It is not clear to the authors how to unify the many derivatives appearing in this document. For example, it is tempting to interpret the finite differences appearing in Main Theorem B as belonging to a kind of pairing:

$$\left\langle \log\left(\frac{x}{\exp_{\mathrm{Td}^\sharp}(x)}\right), t^{-m}x^n \right\rangle = \frac{1}{m \cdot n!} \Delta^m[r^n](0),$$

where the role of t^{-m} is to select the iterated finite difference operator $\Delta^m|_{r=0}$ and x^n to select the test polynomial r^n . This interpretation does not obviously

extend to the positive t -terms appearing in $N_n^{\text{Wit}^\sharp}$, but *some* total interpretation seems surely possible.

Remark 4.4 ([Sof97, Section 3]). Sofer explores a construction which extracts from a Jacobi form a p -adic measure valued in modular forms, using formulas of similar flavor to ours but “with one fewer variable”. It could be interesting to have some identification of the *measure* in Main Theorem A in those terms, or to generalize their construction to produce measures valued in Jacobi forms, or even to see the limitations of any such generalizations.

More concretely, it would be interesting to see topology-free proofs of the congruences described in our Main Theorems. One specific opportunity for improvement is that Ando, Hopkins, and Rezk’s manual analysis allows them to “halve” the Mazur measure [AHRb, Section 10.3]. Figure 3 and Figure 4 suggest that this is possible for the measure constructed in Main Theorem A. Perhaps this is accessible through topology, but it is likely equally accessible directly.

Remark 4.5. The three factors appearing in Lemma 3.2 have discernable proveances in both homotopy theory and in number theory. In homotopy theory, they belong to the finite Adams resolution of the $K(1)$ -local sphere, the Rezk logarithm, and the characteristic series of the Todd orientation. In number theory, they arise from the regularization of a distribution to a measure, the restriction to \mathbb{Z}_p^\times to perform p -adic interpolation, and as special values of the ζ -function. It would be deeply interesting to have a tighter correspondence between these three pieces. The Main Theorems also provide a new proving ground for any such analogy, which should additionally assign a measure-theoretic meaning to $\psi^p(t) = 1 - (1 - t)^p = [p]_{\mathbb{G}_m}(t)$.

Remark 4.6. In its incarnation as an operation on \mathbb{E}_∞ -rings (cf. [CL25]), the sharp construction is part of a more general theory of \mathbb{E}_∞ -orientations of Tate objects. For example, this theory also subsumes the *Frobenius homomorphism* of [NS18]. One wonders what the moment sequence associated to the Frobenius twist of the Witten orientation is.

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	t^0	t^{-1}	t^{-2}	t^{-3}	t^{-4}	t^{-5}	t^{-6}
$M_1^{\text{Td}\#}$... 0000 ₃	... 2220 ₃	... 0000 ₃	... 0010 ₃	... 0100 ₃	... 0200 ₃	... 1000 ₃
$M_2^{\text{Td}\#}$... 1102 ₃	... 2110 ₃	... 0120 ₃	... 1200 ₃	... 2000 ₃	... 1000 ₃	... 1100 ₃
$M_3^{\text{Td}\#}$... 0000 ₃	... 0200 ₃	... 1000 ₃	... 1100 ₃	... 0000 ₃	... 0000 ₃	... 0000 ₃
$M_4^{\text{Td}\#}$... 1022 ₃	... 2120 ₃	... 0010 ₃	... 0200 ₃	... 2200 ₃	... 0000 ₃	... 0000 ₃
$M_5^{\text{Td}\#}$... 0000 ₃	... 1010 ₃	... 1100 ₃	... 1120 ₃	... 2100 ₃	... 2200 ₃	... 0000 ₃
$M_6^{\text{Td}\#}$... 2212 ₃	... 1100 ₃	... 0200 ₃	... 0000 ₃	... 2000 ₃	... 0000 ₃	... 2000 ₃
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$M_{27}^{\text{Td}\#}$... 0000 ₃						
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$M_{30}^{\text{Td}\#}$... 0112 ₃	... 0200 ₃	... 0100 ₃	... 0000 ₃	... 1000 ₃	... 0000 ₃	... 1000 ₃
$M_{31}^{\text{Td}\#}$... 0000 ₃	... 2120 ₃	... 2000 ₃	... 1110 ₃	... 2100 ₃	... 1200 ₃	... 1000 ₃
$M_{32}^{\text{Td}\#}$... 1202 ₃	... 1010 ₃	... 0220 ₃	... 0200 ₃	... 1000 ₃	... 1000 ₃	... 1100 ₃
$M_{33}^{\text{Td}\#}$... 0000 ₃	... 1100 ₃	... 2000 ₃	... 2200 ₃	... 0000 ₃	... 0000 ₃	... 0000 ₃
$M_{34}^{\text{Td}\#}$... 2122 ₃	... 2020 ₃	... 0110 ₃	... 1200 ₃	... 2200 ₃	... 0000 ₃	... 2000 ₃
$M_{35}^{\text{Td}\#}$... 0000 ₃	... 2210 ₃	... 1100 ₃	... 2220 ₃	... 0100 ₃	... 1200 ₃	... 0000 ₃
$M_{36}^{\text{Td}\#}$... 2012 ₃	... 2000 ₃	... 1000 ₃	... 0000 ₃	... 0000 ₃	... 0000 ₃	... 0000 ₃
$M_{37}^{\text{Td}\#}$... 0000 ₃	... 1220 ₃	... 0000 ₃	... 1010 ₃	... 0100 ₃	... 0200 ₃	... 1000 ₃
$M_{38}^{\text{Td}\#}$... 2102 ₃	... 1110 ₃	... 1120 ₃	... 1200 ₃	... 2000 ₃	... 1000 ₃	... 1100 ₃

Figure 1: A range of ternary expansions of the 3-adic Todd[#] moments at $c = 1+p$. Main Theorem B with test polynomial $p^{-i}(r^{(p-1)p^i} - 1)$ shows that the i^{th} digit within a column is periodic with periodicity $(p-1)p^i$. The colors are there to help the eye catch the vertical patterns.

$M_1^{\text{Wit}^\sharp}$	t^6	t^5	t^4	t^3	t^2	t^1
q^1	...2220 ₃	...2210 ₃				
q^2	...2010 ₃	...2020 ₃	...2100 ₃	...2110 ₃	...2200 ₃	...2100 ₃
q^3	...2220 ₃	...2210 ₃				
q^4	...1210 ₃	...2000 ₃	...1120 ₃	...2210 ₃	...2020 ₃	...1110 ₃
q^5	...0120 ₃	...0210 ₃	...1200 ₃	...0220 ₃	...2100 ₃	...1200 ₃
q^6	...2010 ₃	...2020 ₃	...2100 ₃	...2110 ₃	...2200 ₃	...2100 ₃
q^7	...0000 ₃	...0020 ₃	...1110 ₃	...1200 ₃	...2010 ₃	...1020 ₃
q^8	...0020 ₃	...0210 ₃	...2200 ₃	...1020 ₃	...1100 ₃	...2200 ₃
q^9	...2220 ₃	...2210 ₃				

$M_7^{\text{Wit}^\sharp}$	t^6	t^5	t^4	t^3	t^2	t^1
q^1	...2020 ₃	...1110 ₃				
q^2	...1210 ₃	...0120 ₃	...2100 ₃	...1010 ₃	...1200 ₃	...0100 ₃
q^3	...2020 ₃	...1110 ₃				
q^4	...2110 ₃	...1000 ₃	...0220 ₃	...0110 ₃	...2120 ₃	...2010 ₃
q^5	...0220 ₃	...0110 ₃	...1200 ₃	...0020 ₃	...0100 ₃	...0200 ₃
q^6	...1210 ₃	...0120 ₃	...2100 ₃	...1010 ₃	...1200 ₃	...0100 ₃
q^7	...1000 ₃	...1120 ₃	...0010 ₃	...0200 ₃	...1210 ₃	...0120 ₃
q^8	...0120 ₃	...1110 ₃	...2200 ₃	...1120 ₃	...2100 ₃	...1200 ₃
q^9	...2020 ₃	...1110 ₃				

$M_{19}^{\text{Wit}^\sharp}$	t^6	t^5	t^4	t^3	t^2	t^1
q^1	...0220 ₃	...1210 ₃				
q^2	...1010 ₃	...0020 ₃	...2100 ₃	...1110 ₃	...2200 ₃	...2100 ₃
q^3	...0220 ₃	...1210 ₃				
q^4	...0210 ₃	...2000 ₃	...2120 ₃	...1210 ₃	...0020 ₃	...0110 ₃
q^5	...1120 ₃	...2210 ₃	...1200 ₃	...1220 ₃	...2100 ₃	...1200 ₃
q^6	...1010 ₃	...0020 ₃	...2100 ₃	...1110 ₃	...2200 ₃	...2100 ₃
q^7	...0000 ₃	...1020 ₃	...0110 ₃	...1200 ₃	...1010 ₃	...2020 ₃
q^8	...1020 ₃	...2210 ₃	...2200 ₃	...2020 ₃	...1100 ₃	...2200 ₃
q^9	...0220 ₃	...1210 ₃				

Figure 2: Some 3-adic Witten $^\sharp$ moments at $c = 1 + p$, in a range of t - and q -degrees. Congruences here relate the colored digits across different tables rather than within columns. The relevant test polynomials are $p^{-2}(r^{(p-1)p} - 1)$ and $p^{-3}(r^{(p-1)p^2} - 1)$.

	t^0	t^{-1}	t^{-2}	t^{-3}	t^{-4}	t^{-5}
$M_1^{\text{Td}^\sharp}$... 0000000 ₂	... 1111110 ₂	... 1111110 ₂	... 1111100 ₂	... 1111000 ₂	... 1110000 ₂
$M_2^{\text{Td}^\sharp}$... 0101010 ₂	... 1111000 ₂	... 1111000 ₂	... 1100000 ₂	... 0110000 ₂	... 1000000 ₂
$M_3^{\text{Td}^\sharp}$... 0000000 ₂	... 1100110 ₂	... 1100110 ₂	... 1111100 ₂	... 0101000 ₂	... 1100000 ₂
$M_4^{\text{Td}^\sharp}$... 1011010 ₂	... 0110000 ₂	... 0110000 ₂	... 1000000 ₂	... 1100000 ₂	... 0000000 ₂
$M_5^{\text{Td}^\sharp}$... 0000000 ₂	... 0001110 ₂	... 0001110 ₂	... 1111100 ₂	... 1011000 ₂	... 1010000 ₂
$M_6^{\text{Td}^\sharp}$... 0001010 ₂	... 0101000 ₂	... 0101000 ₂	... 0100000 ₂	... 0010000 ₂	... 1000000 ₂
$M_7^{\text{Td}^\sharp}$... 0000000 ₂	... 1110110 ₂	... 1110110 ₂	... 1111100 ₂	... 0001000 ₂	... 1000000 ₂
$M_8^{\text{Td}^\sharp}$... 0111010 ₂	... 1100000 ₂	... 1100000 ₂	... 0000000 ₂	... 1000000 ₂	... 0000000 ₂
$M_9^{\text{Td}^\sharp}$... 0000000 ₂	... 0011110 ₂	... 0011110 ₂	... 1111100 ₂	... 0111000 ₂	... 0110000 ₂
$M_{10}^{\text{Td}^\sharp}$... 1101010 ₂	... 1011000 ₂	... 1011000 ₂	... 1100000 ₂	... 1110000 ₂	... 1000000 ₂
$M_{11}^{\text{Td}^\sharp}$... 0000000 ₂	... 0000110 ₂	... 0000110 ₂	... 1111100 ₂	... 1101000 ₂	... 0100000 ₂
$M_{12}^{\text{Td}^\sharp}$... 0011010 ₂	... 0010000 ₂	... 0010000 ₂	... 1000000 ₂	... 0100000 ₂	... 0000000 ₂
$M_{13}^{\text{Td}^\sharp}$... 0000000 ₂	... 0101110 ₂	... 0101110 ₂	... 1111100 ₂	... 0011000 ₂	... 0010000 ₂
$M_{14}^{\text{Td}^\sharp}$... 1001010 ₂	... 0001000 ₂	... 0001000 ₂	... 0100000 ₂	... 1010000 ₂	... 1000000 ₂
$M_{15}^{\text{Td}^\sharp}$... 0000000 ₂	... 0010110 ₂	... 0010110 ₂	... 1111100 ₂	... 1001000 ₂	... 0000000 ₂
$M_{16}^{\text{Td}^\sharp}$... 1111010 ₂	... 1000000 ₂	... 1000000 ₂	... 0000000 ₂	... 0000000 ₂	... 0000000 ₂
$M_{17}^{\text{Td}^\sharp}$... 0000000 ₂	... 0111110 ₂	... 0111110 ₂	... 1111100 ₂	... 1111000 ₂	... 1110000 ₂
$M_{18}^{\text{Td}^\sharp}$... 0101010 ₂	... 0111000 ₂	... 0111000 ₂	... 1100000 ₂	... 0110000 ₂	... 1000000 ₂
$M_{19}^{\text{Td}^\sharp}$... 0000000 ₂	... 0100110 ₂	... 0100110 ₂	... 1111100 ₂	... 0101000 ₂	... 1100000 ₂
$M_{20}^{\text{Td}^\sharp}$... 1011010 ₂	... 1110000 ₂	... 1110000 ₂	... 1000000 ₂	... 1100000 ₂	... 0000000 ₂
$M_{21}^{\text{Td}^\sharp}$... 0000000 ₂	... 1001110 ₂	... 1001110 ₂	... 1111100 ₂	... 1011000 ₂	... 1010000 ₂
$M_{22}^{\text{Td}^\sharp}$... 0001010 ₂	... 1101000 ₂	... 1101000 ₂	... 0100000 ₂	... 0010000 ₂	... 1000000 ₂
$M_{23}^{\text{Td}^\sharp}$... 0000000 ₂	... 0110110 ₂	... 0110110 ₂	... 1111100 ₂	... 0001000 ₂	... 1000000 ₂
$M_{24}^{\text{Td}^\sharp}$... 0111010 ₂	... 0100000 ₂	... 0100000 ₂	... 0000000 ₂	... 1000000 ₂	... 0000000 ₂
$M_{25}^{\text{Td}^\sharp}$... 0000000 ₂	... 1011110 ₂	... 1011110 ₂	... 1111100 ₂	... 0111000 ₂	... 0110000 ₂
$M_{26}^{\text{Td}^\sharp}$... 1101010 ₂	... 0011000 ₂	... 0011000 ₂	... 1100000 ₂	... 1110000 ₂	... 1000000 ₂
$M_{27}^{\text{Td}^\sharp}$... 0000000 ₂	... 1000110 ₂	... 1000110 ₂	... 1111100 ₂	... 1101000 ₂	... 0100000 ₂
$M_{28}^{\text{Td}^\sharp}$... 0011010 ₂	... 1010000 ₂	... 1010000 ₂	... 1000000 ₂	... 0100000 ₂	... 0000000 ₂
$M_{29}^{\text{Td}^\sharp}$... 0000000 ₂	... 1101110 ₂	... 1101110 ₂	... 1111100 ₂	... 0011000 ₂	... 0010000 ₂
$M_{30}^{\text{Td}^\sharp}$... 1001010 ₂	... 1001000 ₂	... 1001000 ₂	... 0100000 ₂	... 1010000 ₂	... 1000000 ₂
$M_{31}^{\text{Td}^\sharp}$... 0000000 ₂	... 1010110 ₂	... 1010110 ₂	... 1111100 ₂	... 1001000 ₂	... 0000000 ₂
$M_{32}^{\text{Td}^\sharp}$... 1111010 ₂	... 0000000 ₂				
$M_{33}^{\text{Td}^\sharp}$... 0000000 ₂	... 1111110 ₂	... 1111110 ₂	... 1111100 ₂	... 1111000 ₂	... 1110000 ₂
$M_{34}^{\text{Td}^\sharp}$... 0101010 ₂	... 1111000 ₂	... 1111000 ₂	... 1100000 ₂	... 0110000 ₂	... 1000000 ₂
$M_{35}^{\text{Td}^\sharp}$... 0000000 ₂	... 1100110 ₂	... 1100110 ₂	... 1111100 ₂	... 0101000 ₂	... 1100000 ₂
$M_{36}^{\text{Td}^\sharp}$... 1011010 ₂	... 0110000 ₂	... 0110000 ₂	... 1000000 ₂	... 1100000 ₂	... 0000000 ₂

Figure 3: A range of binary expansions of the 2-adic Todd $^\sharp$ moments at $c = 1 + p$. Main Theorem B with test polynomial $2^{-(i+2)}(r^{2^i} - 1)$ shows that the 0th, 1st, and 2nd digit columns are 2-periodic, and then after those the i^{th} digit column is 2^{i-1} -periodic.

$M_1^{\text{Wit}^\#}$	t^6	t^5	t^4	t^3	t^2	t^1
q^1	...111110 ₂	...111100 ₂				
q^2	...111110 ₂	...111100 ₂				
q^3	...000110 ₂	...010100 ₂	...100000 ₂	...101000 ₂	...111000 ₂	...110000 ₂
q^4	...111110 ₂	...111100 ₂				
q^5	...011010 ₂	...000000 ₂	...111100 ₂	...100100 ₂	...110100 ₂	...101000 ₂
q^6	...000110 ₂	...010100 ₂	...100000 ₂	...101000 ₂	...111000 ₂	...110000 ₂

$M_3^{\text{Wit}^\#}$	t^6	t^5	t^4	t^3	t^2	t^1
q^1	...100110 ₂	...001100 ₂				
q^2	...100110 ₂	...001100 ₂				
q^3	...001110 ₂	...110100 ₂	...110000 ₂	...011000 ₂	...101000 ₂	...010000 ₂
q^4	...100110 ₂	...001100 ₂				
q^5	...110010 ₂	...110000 ₂	...011100 ₂	...100100 ₂	...110100 ₂	...101000 ₂
q^6	...001110 ₂	...110100 ₂	...110000 ₂	...011000 ₂	...101000 ₂	...010000 ₂

$M_5^{\text{Wit}^\#}$	t^6	t^5	t^4	t^3	t^2	t^1
q^1	...001110 ₂	...011100 ₂				
q^2	...001110 ₂	...011100 ₂				
q^3	...010110 ₂	...010100 ₂	...000000 ₂	...001000 ₂	...011000 ₂	...110000 ₂
q^4	...001110 ₂	...011100 ₂				
q^5	...001010 ₂	...100000 ₂	...111100 ₂	...100100 ₂	...110100 ₂	...101000 ₂
q^6	...010110 ₂	...010100 ₂	...000000 ₂	...001000 ₂	...011000 ₂	...110000 ₂

$M_9^{\text{Wit}^\#}$	t^6	t^5	t^4	t^3	t^2	t^1
q^1	...011110 ₂	...111100 ₂				
q^2	...011110 ₂	...111100 ₂				
q^3	...100110 ₂	...010100 ₂	...100000 ₂	...101000 ₂	...111000 ₂	...110000 ₂
q^4	...011110 ₂	...111100 ₂				
q^5	...111010 ₂	...000000 ₂	...111100 ₂	...100100 ₂	...110100 ₂	...101000 ₂
q^6	...100110 ₂	...010100 ₂	...100000 ₂	...101000 ₂	...111000 ₂	...110000 ₂

Figure 4: Some 2-adic Witten $^\#$ moments at $c = 1 + p$, in a range of t - and q -degrees. Congruences here relate the colored digits across different tables rather than within columns. Congruences here relate the colored digits across different tables rather than within columns. The relevant test polynomials are $p^{-3}(r^2 - 1)$, $p^{-4}(r^4 - 1)$, and $p^{-5}(r^8 - 1)$.