MATH 55: HOMEWORK #2 SOLUTIONS

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1. SECTION 1.5: NESTED QUANTIFIERS

1.1. Problem 1.5.28. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers:

(1) $\forall x \exists y (x^2 = y)$: This is true; the rule $y = x^2$ determines a function, and hence the quantity y exists for any x.

(2) $\forall x \exists y (x = y^2)$: This is not true; when $x = -1$, there is no y with $y^2 = x = -1$.

(3) $\exists x \forall y (xy = 0)$: This is true; $x = 0$ is a witness, as it has the absorptive property $xy = 0 \cdot y = 0$.

(4) $\exists x \exists y (x + y \neq y + x)$: This is false; addition of real numbers is commutative.

(5) $\forall x (x \neq 0 \rightarrow \exists y (xy = 1))$: This is true; for any nonzero x, the assignment $y = 1/x$ has the property $xy = x \cdot (1/x) = 1$.

(6) $\exists x \forall y (y \neq 0 \rightarrow xy = 1)$: This is false. If it were true, then there would be an x for which the proposition is true for $y = 2$ and 3 simultaneously, i.e., $2x = 1$ and $3x = 1$. Taking the difference of these two equations, we find
$$x = 3x - 2x = 1 - 1 = 0,$$
but $x \cdot y = 0 \cdot y = 0$ for any y, making our premise absurd.

(7) $\forall x \exists y (x + y = 1)$: This is true; set $y = 1 - x$.

(8) $\exists x \exists y (x + 2y = 2 \wedge 2x + 4y = 5)$: This is false. Divide the second equation by 2 to get the equivalent equation $x + 2y = 5/2$. The line it determines is of the same slope as the first equation, but they have distinct y-intercepts, which means that they are parallel and hence never intersect. So, they cannot have a common solution (x, y).

(9) $\forall x \exists y (x + y = 2 \wedge 2x - y = 1)$: This is false. The linear system has the unique solution $(x, y) = (1, 1)$, which means that for the choice of (say) $x = 3$, there there does *not* exist a choice of y satisfying both $3 + y = 2$ (which requires $y = -1$) and $4 - y = 1$ (which requires $y = 3$).

(10) $\forall x \forall y \exists z (z = (x + y)/2)$: This is true; the rule $z = (x + y)/2$ determines a function, and hence the quantity z exists for any x and y.

1.2. Problem 1.5.30. Rewrite each of these statements so that negations appear only within predicates.

(1) $\neg \exists y \exists x P(x, y)$:
$$\neg \exists y \exists x P(x, y) \equiv \forall y \neg \exists x P(x, y)$$
$$\equiv \forall y \exists y \neg P(x, y).$$

(2) $\neg \forall x \exists y P(x, y)$:
$$\neg \forall x \exists y P(x, y) \equiv \exists x \neg \exists y P(x, y)$$
$$\equiv \exists x \forall y \neg P(x, y).$$

(3) $\neg \exists y (Q(y) \wedge \forall x \neg R(x, y))$:
$$\neg \exists y (Q(y) \wedge \forall x \neg R(x, y)) \equiv \forall y \neg (Q(y) \wedge \forall x \neg R(x, y))$$
$$\equiv \forall y (\neg Q(y) \vee \neg \forall x \neg R(x, y))$$
$$\equiv \forall y (\neg Q(y) \vee \exists x R(x, y)).$$

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(4) $\neg\exists y(\exists x R(x,y) \lor \forall x S(x,y))$:

$$\neg\exists y(\exists x R(x,y) \lor \forall x S(x,y)) \equiv \forall y\neg(\exists x R(x,y) \lor \forall x S(x,y))$$
$$\equiv \forall y(\neg\exists x R(x,y) \land \neg\forall x S(x,y))$$
$$\equiv \forall y(\forall x\neg R(x,y) \land \exists x\neg S(x,y)).$$

(5) $\neg\exists y(\forall x\exists z T(x,y,z) \lor \exists x\forall z U(x,y,z))$:

$$\neg\exists y(\forall x\exists z T(x,y,z) \lor \exists x\forall z U(x,y,z)) \equiv \forall y\neg(\forall x\exists z T(x,y,z) \lor \exists x\forall z U(x,y,z))$$
$$\equiv \forall y(\neg\forall x\exists z T(x,y,z) \land \neg\exists x\forall z U(x,y,z))$$
$$\equiv \forall y(\exists x\neg\exists z T(x,y,z) \land \forall x\neg\forall z U(x,y,z))$$
$$\equiv \forall y(\exists x\forall z\neg T(x,y,z) \land \forall x\exists z\neg U(x,y,z)).$$

1.3. **Problem 1.5.34.** Find a common domain for the variables x, y, and z for which the statement

$$\forall x\forall y((x \neq y) \to \forall z((z = x) \lor (z = y)))$$

is true and another domain for which it is false.

Solution: On the domain $D = \{a,b\}$ (or $D = \{a\}$, or $D = \varnothing$), this statement is true. To prove the implication, we check that the predicate is true in the case that the antecedent is true. If x and y are distinct elements of the domain, as indicated by the antecedent, then one must be a and the other b. Then, any third choice of element z of the domain must also either be equal to a or b, and hence to one of x and y, proving the claim. However, on a domain with more than two elements the claim is false. Set $D = \{a,b,c\}$. We exhibit a counterexample by setting $x = a$, $y = b$, $z = c$. It is true that $x \neq y$, but false that $(z = x) \lor (z = y)$, and hence the implication is false.

1.4. **Problem 1.5.40.** Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

(1) $\forall x\exists y(x = 1/y)$: This is not true when quantified over the integers. Setting $x = 2$, there is no integer "$1/2$" with the property $xy = 1$. (For the curious, this follows from order properties: since 1 is the smallest positive integer, multiplication by any y with $|y| > 1$ has the property that $|x| < |xy|$. We can check that $y = -1$, $y = 0$, and $y = 1$ each do not have the property that $xy = 1$, and by this other fact, neither can any y with $|y| > 1$, since then $2 = |x| < |xy|$, whereas $|1| = 1$.)

(2) $\forall x\exists y(y^2 - x < 100)$: This is also false. Set $x = -100$; then as y^2 is a nonnegative value, $y^2 - x = y^2 + 100$ will always be at least 100, hence never less than 100.

(3) $\forall x\forall y(x^2 \neq y^3)$: This, too, is false. For instance, set $x = y = 1$; then $x^2 = 1 = y^3$. (More generally, n^6 for any n has both a square- and cube-root, giving candidates for x and y respectively.)

1.5. **Problem 1.5.52.** Express the quantification $\exists! x P(x)$, introduced in Section 1.4, using universal quantifications, existential quantifications, and logical operators.

$$(\exists! x P(x)) \equiv ((\exists x P(x)) \land (\forall x\forall y((P(x) \land P(y)) \to (x = y))).$$

2. 1.6: RULES OF INFERENCE

2.1. **Problem 1.6.12.** Show that the argument form with premises $(p \land t) \to (r \lor s)$, $q \to (u \land t)$, $u \to p$, and $\neg s$ and conclusion $q \to r$ is valid first by checking a truth-table and then by using the rules of inference.

Solution: Because we are checking the validity of an implication, we need only check that when its antecedent is true (q), then so is its predicate (r). By assumption, all the premises are valid implications, and hence if q is true, then the second premise requires that $u \land t$ be true, i.e., that u is true and t is true. If u is true, then the third premise implies that p is true. Since both p and t are true, the first premise implies that $r \lor s$ is true. Since the final premise asserts that $\neg s$ is true, it must be that r is true, which is the desired conclusion.

In terms of inference rules, we can build the following proof tree:

$$\dfrac{\dfrac{q \to u \land t \quad \dfrac{\overline{u \to p}}{u \land t \to p}}{\dfrac{q \to p \land t}{\dfrac{q \to r \lor s}{\therefore q \to r.}} \quad p \land t \to r \lor s \quad \overline{\neg s}}}{}$$

2.2. **Problem 1.6.24.** Identify the error or errors in this argument that supposedly shows that if $\forall x(P(x) \lor Q(x))$ is true that $\forall x P(x) \lor \forall x Q(x)$ is true:

1. $\forall x(P(x) \lor Q(x))$ Premise
2. $P(c) \lor Q(c)$ Universal instantiation from 1
3. $P(c)$ Simplification from 2
4. $\forall x P(x)$ Universal generalization from 3
5. $Q(c)$ Simplification from 2
6. $\forall x Q(x)$ Universal generalization from 5
7. $\forall x(P(x) \lor \forall x Q(x))$ Conjunction from 4 and 6.

Solution: There are three errors. First, simplification is the rule $(p \land q) \to p$; the rule does not apply to $\lor$. This causes an error on lines 3 and 5. Finally, the conjunction rule is misapplied on the final line — look closely at the placement of the parentheses.

2.3. **Problem 1.6.28.** Use rules of inference to show that if $\forall x(P(x) \lor Q(x))$ and $\forall x((\neg P(x) \land Q(x)) \to R(x))$ are true, then $\forall x(\neg R(x) \to P(x))$ is also true, where the domains of all quantifiers are the same.

$$\dfrac{\dfrac{\dfrac{\overline{\forall x(P(x) \lor Q(x))}}{\dfrac{P(c) \lor Q(c)}{\neg P(c) \to Q(c)}} \quad \dfrac{\overline{\forall x((\neg P(x) \land Q(x)) \to R(x))}}{\neg P(c) \land Q(c) \to R(c)}}{\dfrac{\neg P(c) \to R(c)}{\neg R(c) \to P(c) \text{ for a general } c}}}{\therefore \forall x(\neg R(x) \to P(x)).}$$

2.4. **Problem 1.6.34.** The Logic Problem, taken from *WFF'N PROOF, The Game of Logic*, has these two assumptions:

(1) "Logic is not difficult or not many students like logic."
(2) "If mathematics is easy, then logic is not difficult."

We will denote by p the proposition "logic is not difficult", by q the proposition "not many students like logic", and by r the proposition "mathematics is easy". The premises then take the form $r \to p$ and $p \lor q$ (which is equivalent to $\neg p \to q$). Determine whether each of the following are valid conclusions of these assumptions:

(1) "That mathematics is not easy, if many students like logic." This is equivalent to $\neg q \to \neg r$. This is a valid syllogysm of $\neg p \to \neg r$ and $\neg q \to \neg p$, using the contrapositives of the premises.
(2) "That not many students like logic, if mathematics is not easy." This is equivalent to $\neg r \to q$, which is not a valid syllogysm of the above premises.
(3) "That mathematics is not easy or logic is difficult." This is equivalent to $\neg r \lor \neg p \equiv r \to \neg p$. This is the negation of one of the above premises and not a valid syllogysm of them.
(4) "That logic is not difficult or mathematics is not easy." This is equivalent to $p \lor \neg r \equiv r \to p$, which is one of the above premises.
(5) "That if not many students like logic, then either mathematics is not easy or logic is not difficult." This is equivalent to $q \to (\neg r \lor p)$, which is valid. Namely, the conclusion $\neg r \lor p$ can be interpreted as the implication $r \to p$, which is one of our hypotheses. Conjunction says that $q \land (r \to p)$ follows from $r \to p$ alone.

3.1. Problem 1.7.8. Prove that if n is a perfect square, then $n+2$ is not a perfect square. *Proof:* Suppose that $n = x^2$ and $n + 2 = y^2$, with both x and y nonnegative naturals. We know a formula for the difference of squares:

$$(n+2) - (n) = x^2 - y^2$$
$$2 = (x-y)(x+y).$$

Since 2 is a prime number, one of these factors must be 2 and the other must be 1. Since both x and y are assumed nonnegative, it must be that $x + y > x - y$, so $x - y = 1$ and $x + y = 2$. This system does not have a solution over the naturals: the first equation begets the substitution $x = 1 + y$, which transforms the second into $x + y = (1+y) + y = 1 + 2y = 2$. Whenever y is an integer, this middle expression is an odd number, and hence can never be equal to 2. So, we must have been wrong all along: there can be no pair of perfect squares whose difference is two.

Another argument (which is also a proof by contradiction) proceeds by noting that if $n = x^2$ and $n+2 = y^2$, with both x and y nonnegative naturals, then $n + 2 \geq (x+1)^2$, because any perfect square greater than x^2 must be at least $(x+1)^2$. This gives

$$n + 2 \geq (x+1)^2 = x^2 + 2x + 1 = n + 2x + 1,$$

and hence $2x + 1 \leq 2$. Because x is a natural number, this is only possible if $x = 0$ and hence $n = 0$, but in this case we have $2 = n + 2 = y^2$, which is impossible. So, again we must have been wrong all along: there can be no pair of perfect squares whose difference is two.

3.2. Problem 1.7.16. Prove that if m and n are integers and mn is even, then m is even or n is even. *Proof:* Suppose that m and n are both odd; we will work to show that mn is also odd. Suppose first that it's not. In that case, 2 divides it, and because 2 is a prime number, it must divide at least one of the two factors m and n. However, these were assumed to be odd (i.e., indivisible by 2), and so this cannot be the case. Hence, mn must be odd. (Note: you don't *need* to go through contradiction to prove this statement, but it does make it shorter. You're encouraged to write a proof that doesn't use contradiction to see that it can be done.)

3.3. Problem 1.7.30. Show that these three statements are equivalent, where a and b are real numbers:
(1) a is less than b.
(2) The average of a and b is greater than a.
(3) The average of a and b is less than b.

Proof: We will show that 1 and 2 are equivalent, then that 1 and 3 are equivalent. Consider the following sequence of inequalities:

$$a < b \qquad \text{(Initial assumption.)}$$
$$a + a < a + b \qquad \text{(Add } a \text{ to both sides.)}$$
$$\frac{a+a}{2} < \frac{a+b}{2} \qquad \text{(Divide both sides by 2.)}$$
$$a < \frac{a+b}{2}. \qquad \text{(Simplify the left.)}$$

This shows that 1 implies 2. To show that 2 implies 1, read the same proof backwards — all the steps involved are reversible. The equivalence between 1 and 3 can be shown be a similar sequence:

$$a < b \qquad \text{(Initial assumption.)}$$
$$a + b < b + b \qquad \text{(Add } b \text{ to both sides.)}$$
$$\frac{a+b}{2} < \frac{b+b}{2} \qquad \text{(Divide both sides by 2.)}$$
$$\frac{a+b}{2} < b. \qquad \text{(Simplify the right.)}$$

3.4. Problem 1.7.36. Show that the propositions p_1, p_2, p_3, and p_4 can be shown to be equivalent by showing that $p_1 \leftrightarrow p_4$, $p_2 \leftrightarrow p_3$, and $p_1 \leftrightarrow p_3$.

Proof: Biimplications have the property that $a \leftrightarrow b$ and $b \leftrightarrow c$ imply $a \leftrightarrow c$, and that $a \leftrightarrow b$ is equivalent to $b \leftrightarrow a$. To show that all these propositions are equivalent, we need to check that $p_i \leftrightarrow p_j$ for each choice of $i < j$.

- $p_1 \leftrightarrow p_2$: This follows from $p_1 \leftrightarrow p_3$, the last premise, and $p_3 \leftrightarrow p_2$, the second premise.
- $p_1 \leftrightarrow p_3$: This is the third premise.
- $p_1 \leftrightarrow p_4$: This is the first premise.
- $p_2 \leftrightarrow p_3$: This is the second premise.
- $p_2 \leftrightarrow p_4$: This follows from joining $p_2 \leftrightarrow p_1$ (see earlier in this list) with the first premise $p_1 \leftrightarrow p_4$ to form $p_2 \leftrightarrow p_4$.
- $p_3 \leftrightarrow p_4$: This follows from joining the last premise $p_3 \leftrightarrow p_1$ with the first premise $p_1 \leftrightarrow p_4$ to form $p_3 \leftrightarrow p_4$.

(Note: In general, draw a picture with a dot representing each proposition p_i and a line connecting the dot p_i to the dot p_j for each premise $p_i \leftrightarrow p_j$. If the picture cannot be separated into two connected parts, then all the propositions are equivalent.)

4. 1.8: Proof Methods and Strategy

4.1. Problem 1.8.4. Use a proof by cases to show that $\min(a, \min(b, c)) = \min(\min(a, b), c)$ whenever a, b, and c are real numbers. *Proof:* There are six cases.

- $a \leq b \leq c$: $\min(a, \min(b, c)) = \min(a, b) = a$. $\min(\min(a, b), c) = \min(a, c) = a$.
- $a \leq c \leq b$: $\min(a, \min(b, c)) = \min(a, c) = a$. $\min(\min(a, b), c) = \min(a, c) = a$.
- $b \leq a \leq c$: $\min(a, \min(b, c)) = \min(a, b) = b$. $\min(\min(a, b), c) = \min(b, c) = b$.
- $b \leq c \leq a$: $\min(a, \min(b, c)) = \min(a, b) = b$. $\min(\min(a, b), c) = \min(b, c) = b$.
- $c \leq a \leq b$: $\min(a, \min(b, c)) = \min(a, c) = c$. $\min(\min(a, b), c) = \min(a, c) = c$.
- $c \leq b \leq a$: $\min(a, \min(b, c)) = \min(a, c) = c$. $\min(\min(a, b), c) = \min(b, c) = c$.

4.2. Problem 1.8.10. Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square. Is your proof constructive or nonconstructive? *Proof:* We can re-use the proof given earlier in these solutions. Assume that they are both perfect squares: then the first can be expressed as x^2 and the second as y^2 for distinct positive integers x and y. We then consider their difference:

$$(2 \cdot 10^{500} + 16) - (2 \cdot 10^{500} + 15) = y^2 - x^2$$
$$1 = (y - x)(y + x).$$

Since the only positive factors of 1 are 1 itself, it must simultaneously be the case that $y - x = 1$ and $y + x = 1$. This is not possible, and hence both numbers cannot be perfect squares. This proof is nonconstructive — a constructive proof would produce a factorization of one of the numbers and check that it, indeed, is not a perfect square. Mathematica helpfully provided such a factorization of $2 \cdot 10^{500} + 16$:

$2 \cdot 10^{500} + 16 = 2^4 \cdot 3^3 \cdot 99613592233^1 \cdot 41347646167753^1 \cdot 1124027291138587276289828907422751349800788 \cdots$
$\cdots 6094747850774314850558938752114094414571887222131729481977403358237747556610488 \cdots$
$\cdots 4784252688332947800710132821600289835943764379665940535551467647927795402345 41 \cdots$
$\cdots 4456622418363208484780664887497261965943655207576206742868626015313631142493 45 \cdots$
$\cdots 1771315599693084257619264349054297089460394898960482026247592074968018934358 18 \cdots$
$\cdots 7762672445998143509575281708529794548428680131553313642918896849579622881694 28 \cdots$
$\cdots 0648335872654019547420854237438021756518 7^1.$

Since not all the factors appear an even number of times (3^3, for instance), this number is not a perfect square.

4.3. **Problem 1.8.26.** Suppose that five ones and four zeroes are arranged around a circle. Between any two equal bits you insert a 0 and and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeroes. *Proof:* To ease typesetting, I'll write strings of numbers rather than loops of them. The idea is to construct what the previous stage must have been, given the result of such an iteration. Suppose that we've ended up the string 000000000. In this case, the previous string must have all consisted of identical numbers, i.e., either 000000000 or 111111111. We are already in the process of analyzing the first case, so we will turn our attention to the second. In this case, the string preceding 111111111 must have consisted entirely of alternating digits. In particular, there must have been some digit i which was a 1 — we will write this as our first digit, which is allowed because we can always rotate the ring of numbers (and hence the digits in our string). The string must then look like: 101010101 — but, up to rotation, this is the same string as 010101011. Since the last two digits are identical, it's not possible to form a string with all alternating digits to produce 111111111. Hence, the only way to produce the string 000000000 is to begin with it or with 111111111 — there can be no further history. Since we begin with a nonzero number of both 0 and 1 digits, we cannot be in this case, and hence cannot reach 000000000.