

MATH 55: HOMEWORK #7 SOLUTIONS

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1. SECTION 5.2: STRONG INDUCTION AND WELL-ORDERING

1.1. Problem 5.2.4. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

- (1) Show statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of the proof.
 - (a) $P(18)$: 18 can be written as $7 + 7 + 4 = 7 \cdot 2 + 4 \cdot 1$.
 - (b) $P(19)$: 19 can be written as $7 + 4 + 4 + 4 = 7 \cdot 1 + 4 \cdot 3$.
 - (c) $P(20)$: 20 can be written as $4 + 4 + 4 + 4 + 4 = 7 \cdot 0 + 4 \cdot 5$.
 - (d) $P(21)$: 21 can be written as $7 + 7 + 7 = 7 \cdot 3 + 4 \cdot 0$.
- (2) What is the inductive hypothesis of the proof?

Let n satisfy $n \geq 22$, and suppose that $P(k)$ is true for each $18 \leq k < n$.

- (3) What do you need to prove in the inductive step?

Show that $P(n)$ is true.

- (4) Complete the inductive step for $k > 21$.

If $P(k)$ is true for each $18 \leq k < n$, then in particular $P(n-4)$ is true. Given that, we see that there are numbers $a, b \in \mathbb{Z}_{\geq 0}$ with $4a + 7b = n-4$. Adding 4 to both sides, we have $4(a+1) + 7b = n$, and hence that $P(n)$ is true.

- (5) Explain why these steps show that this statement is true whenever $n \geq 18$.

The *principle of strong induction* collects these facts together to guarantee that $P(n)$ is true for any $n \geq 18$. Literally:

$$\text{StrongInduction} \frac{P(18) \quad P(19) \quad P(20) \quad P(21) \quad \forall(n \geq 22)((\forall(18 \leq k < n)P(k)) \rightarrow P(n))}{\forall(18 \leq n)P(n)}.$$

In words, if we're interested in the truth value of $P(n)$ for any particular $n \geq 18$, then the implication assumption shows that to force $P(n)$ to be true, it suffices to check that $P(k)$ is true for all k in the range $18 \leq k < n$. (Indeed, the proof relies just on the truth of $P(n-4)$, but there's no harm in making this stronger assumption.) After iterating this same reduction ($n-23$) times, we will have reduced the problem to deciding the truth values of $P(18)$ through $P(21)$ — and these we have demonstrated manually.

Alternatively, our proof of this problem can be thought of as four classical mathematical inductions, running simultaneously on the residue classes in \mathbb{Z}_4 . Given any particular integer $n \geq 18$, it is congruent to one of 2, 3, 0, or 1 modulo 4. In each of these cases, either $n-18$, $n-19$, $n-20$, or $n-21$ respectively is nonnegative and divisible by 4. This indicates that we can each the case of $P(n)$ by beginning at $P(18)$, $P(19)$, $P(20)$, or $P(21)$ respectively and applying $P(k-4) \rightarrow P(k)$ to skip ahead in increments of 4 to reach $P(n)$. (Formally, one would set $Q(n) = P(4n-2) \wedge P(4n-1) \wedge P(4n) \wedge P(4n+1)$, check $Q(n) \rightarrow Q(n+1)$, and check that $\forall n Q(n)$ implies $\forall n P(n)$.)

This moral explanation is near to a proof of the principle of strong induction from the well-ordering of $\mathbb{Z}_{\geq 18}$, but it is not a full proof. It also seems that a proof does not appear in the book, so we may as well record one here. Firstly, given any proof by induction, it can be made into a proof by strong induction simply by inserting the word “strong” — this gives extra inductive hypotheses (which the

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proof will not make use of) and does not change the truth value of the inductive implication. To show that strong induction reduces to induction, consider the proposition $Q(n) = \bigwedge_{i=1}^n P(n)$. Then:

$$\begin{array}{c}
 \frac{\text{Induction} \quad \frac{\text{Conjunctive Syllogism} \quad \frac{\begin{array}{c} \forall(n \geq 1)((\forall(1 \leq k \leq n)P(k)) \rightarrow P(n+1)) \\ \forall(n \geq 1)(Q(n) \rightarrow P(n+1)) \end{array}}{\forall(n \geq 1)(Q(n) \rightarrow (Q(n) \wedge P(n+1)))} \quad \forall(n \geq 1)(Q(n) \rightarrow Q(n+1))}{\forall(n \geq 1)Q(n)}.
 \end{array}$$

Hence, if induction is a valid inference rule, then so is strong induction.

1.2. Problem 5.2.6.

(1) Determine which amounts of postage can be formed using just 3-cent and 10-cent stamps.
 Let $P(n)$ denote the proposition “ n cents can be expressed as a sum of 3- and 10-cent pieces”.
 Here are a table of the truth values of $P(n)$ for the first 20 inputs:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20.

Just as in the previous exercise, once we see three in a row at 18, 19, and 20, we will find that every value above 20 is expressable.

(2) Prove your answer to the first part using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

We will separately check the low ranges: $3 = 3 \cdot 1$, $6 = 3 \cdot 2$, $9 = 3 \cdot 3$, $10 = 10 \cdot 1$, $12 = 3 \cdot 4$, $13 = 10 \cdot 1 + 31$, $15 = 3 \cdot 5$, and $16 = 10 \cdot 1 + 3 \cdot 2$ form an exhaustive list of the available combinations in the range $1, \dots, 17$.

The rest of the values we will handle by induction. Let $Q(n)$ denote the conjunction $Q(n) = P(3n) \wedge P(3n+1) \wedge P(3n+2)$. We demonstrate the base case $Q(6) = P(18) \wedge P(19) \wedge P(20)$. These conjunctive terms are witnessed to be true by the combinations $18 = 3 \cdot 6$, $19 = 10 \cdot 1 + 3 \cdot 3$, and $20 = 10 \cdot 2$. In the inductive step, we will show that $Q(n-1)$ implies $Q(n)$. Suppose that $Q(n-1)$ is true; then there must exist $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{Z}_{\geq 0}$ satisfying:

$$\begin{aligned}3a_1 + 10b_1 &= 3n - 3, \\3a_2 + 10b_2 &= 3n - 2, \\3a_3 + 10b_3 &= 3n - 1.\end{aligned}$$

Adding 3 to both sides of each of these equations, we find

$$\begin{aligned}3(a_1 + 1) + 10b_1 &= 3n, \\3(a_2 + 1) + 10b_2 &= 3n + 1, \\3(a_3 + 1) + 10b_3 &= 3n + 2.\end{aligned}$$

and hence $Q(n)$ is true. By induction, $\forall(n \geq 6)Q(n) \equiv \forall(n \geq 6)(P(3n) \wedge P(3n + 1) \wedge P(3n + 2))$ is true. As in the previous exercise, this entails $\forall(n \geq 18)P(n)$.

(3) Prove your answer to the first part using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

Just as in the previous proof, we manually prove the cases 1 through 17. Then, let $R(n)$ denote the proposition that $P(k)$ is true for all $18 \leq k \leq n$. To form the basis of the induction, we check $R(18)$, $R(19)$, and $R(20)$ are true, also as in the previous proof. Finally, we must show that $R(n)$ for $n > 20$ implies $P(n+1)$ (and hence $R(n+1)$): if $R(n)$ is true, then $\forall(18 \leq k \leq n)P(k)$ is true, and in particular, $P(n-2)$ is true. Since $P(n-2)$ is true, there exist $a, b \in \mathbb{Z}_{\geq 0}$ satisfying $3a + 10b = n-2$. Adding 3 to both sides, we have $3(a+1) + 10b = n+1$, and hence conclude that $P(n+1)$ — and $R(n+1)$ — is true.

1.3. Problem 5.2.10. Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, a smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

It always takes $(n - 1)$ snaps to completely break a chocolate bar up — 3 for a four-piece Kit-Kat and 11 for a twelve-piece Hershey's bar. This is shown inductively as follows: let $c(n)$ denote the number of snaps required to break a rectangular bar with n components into n pieces. For a rectangular bar containing two sub-pieces, one snap is required to separate them. Hence, $c(2) = 1$. Then, for a larger bar with $n + 1$ pieces, any snap will break the $n + 1$ pieces into two halves: one piece with k pieces, $0 < k < n + 1$, and the other half with $n + 1 - k$ pieces. Hence, $c(n + 1) = c(k) + c(n + 1 - k) + 1$, where the additional term of 1 comes from the snap we just performed. By the strong inductive hypothesis, since k and $n + 1 - k$ are both less than $n + 1$, we know that the formula holds for them:

$$c(n + 1) = c(k) + c(n + 1 - k) + 1 = (k - 1) + (n + 1 - k - 1) + 1 = n.$$

Hence, the formula holds for $c(n + 1)$ — regardless of how k is chosen — and so the inductive conclusion is shown. The principle of strong induction shows that the formula holds for every choice of n .

1.4. Problem 5.2.14. Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs . Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n - 1)/2$.

We argue by induction. In the base case of $n = 2$, the stones are split into two piles of one stone each. Collecting two piles of one stone each together emits the number 1, which agrees with the formula's predicted value of $2(2 - 1)/2 = 1$. Now suppose that we are playing this game with n stones, and that we are on the last step: there are two remaining piles, one with k stones and one with $n - k$ stones. So far, the process of gathering the k stones together has emitted various numbers, which by inductive hypothesis have summed together to $k(k - 1)/2$. Similarly, the process of gathering the $n - k$ other stones together has emitted various other numbers, which by inductive hypothesis have summed together to $(n - k)(n - k - 1)/2$. Performing the final collection step emits the number $k(n - k)$, which altogether sums to

$$\frac{k(k - 1)}{2} + \frac{(n - k)(n - k - 1)}{2} + k(n - k) = \frac{k^2 - k + n^2 - 2nk + k^2 - n + k + 2nk - 2k^2}{2} = \frac{n^2 - n}{2} = \frac{n(n - 1)}{2},$$

as desired. Hence, the formula holds for n stones if it holds for k stones for all $k < n$, and by the principle of strong induction, it therefore holds for all numbers of stones, unconditionally.

1.5. Problem 5.2.34. Prove that $\sum_{j=1}^n j(j + 1)(j + 2) \cdots (j + k - 1) = n(n + 1)(n + 2) \cdots (n + k)/(k + 1)$ for all positive integers k and n . (Hint: use right-ward, up-ward, or bi-directional induction.)

We re-express this formula as

$$\sum_{j=1}^n \prod_{i=0}^{k-1} (j + i) = \frac{1}{k + 1} \prod_{i=0}^k (n + i).$$

Fixing k , we will prove this by induction on n . When $n = 1$, the two products are identical:

$$\begin{aligned} \sum_{j=1}^1 \prod_{i=0}^{k-1} (j + i) &= \frac{1}{k + 1} \prod_{i=0}^k (1 + i) \\ \prod_{i=0}^{k-1} (1 + i) &= \prod_{i=0}^{k-1} (1 + i), \end{aligned}$$

and hence the base case is true. Now suppose that it is true for a fixed n ; we will add a product to both sides to deduce the correspondence statement for $n + 1$:

$$\begin{aligned}
\sum_{j=1}^n \prod_{i=0}^{k-1} (j+i) &= \frac{1}{k+1} \prod_{i=0}^k (n+i) \\
\left(\sum_{j=1}^n \prod_{i=0}^{k-1} (j+i) \right) + \prod_{i=0}^{k-1} (n+1+i) &= \left(\frac{1}{k+1} \prod_{i=0}^k (n+i) \right) + \prod_{i=0}^{k-1} (n+1+i) \\
\sum_{j=1}^{n+1} \prod_{i=0}^{k-1} (j+i) &= \frac{1}{k+1} \left(\prod_{i=0}^k (n+i) + (k+1) \prod_{i=1}^k (n+i) \right) \\
&= \frac{1}{k+1} \cdot \prod_{i=1}^k (n+i) \cdot ((n+0) + (k+1)) \\
&= \frac{1}{k+1} \prod_{i=1}^{k+1} (n+i) \\
&= \frac{1}{k+1} \prod_{i=0}^k ((n+1)+i).
\end{aligned}$$

By induction on n , it is true for all n for any given k , and hence true for all positive values of k and n .

2. SECTION 5.3: RECURSIVE DEFINITIONS AND STRUCTURAL INDUCTION

2.1. Problem 5.3.12. Let f_n denote the n th Fibonacci number. Prove that $f_1^2 + \cdots + f_n^2 = f_n f_{n+1}$.

The Fibonacci numbers are the sequence satisfying $f_1 = 1$, $f_2 = 1$, and $f_{n+2} = f_n + f_{n+1}$. The identity is true for $n = 1$ and $n = 2$: $1^2 = 1 \cdot 1$ and $1^2 + 1^2 = 1 \cdot 2$. Beyond this, we use induction: suppose that

$$f_1^2 + \cdots + f_n^2 = f_n f_{n+1}$$

is true for a given n . Adding f_{n+1}^2 to both sides, we produce the identity

$$f_1^2 + \cdots + f_n^2 + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2.$$

The right-hand side factors as $(f_n + f_{n+1})f_{n+1}$, and we recognize the left factor as $f_n + f_{n+1} = f_{n+2}$. Putting this together, we have

$$f_1^2 + \cdots + f_{n+1}^2 = f_{n+1} f_{n+2},$$

and hence the identity is true for the case of $n + 1$. Finally, induction shows that the identity is true in all cases.

2.2. Problem 5.3.14. For f_n as before, show that $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$ when n is a positive integer.

Again, we proceed by induction. When $n = 2$, we have $f_3 f_1 - f_2^2 = 2 \cdot 1 - 1^2 = 1$, which agrees with $(-1)^2$. To perform the inductive step, suppose that the formula holds for a given n ; we'll manipulate the equation to show that it holds for $n + 1$ as well. Recall that the definition of f_{n+1} is $f_{n+1} = f_n + f_{n-1}$. This can be rearranged to $f_{n-1} = f_{n+1} - f_n$. Then:

$$\begin{aligned}
f_{n+1} f_{n-1} - f_n^2 &= (-1)^n \\
f_{n+1} (f_{n+1} - f_n) - f_n^2 &= (-1)^n \\
f_{n+1}^2 - (f_{n+1} + f_n) f_n &= (-1)^n \\
f_{n+1}^2 - f_{n+2} f_n &= (-1)^n \\
f_{n+2} f_n - f_{n+1}^2 &= (-1)^{n+1},
\end{aligned}$$

as desired.

2.3. **Problem 5.3.20.** Give a recursive definition of the functions \max and \min so that $\max(a_1, \dots, a_n)$ and $\min(a_1, \dots, a_n)$ are the maximum and minimum of the n numbers a_1, \dots, a_n respectively.

Here are our definitions:

$$\begin{aligned}\max(a_1) &= a_1, & \min(a_1) &= a_1, \\ \max(a_1, a_2) &= \begin{cases} a_1 & \text{if } a_1 \geq a_2, \\ a_2 & \text{if } a_2 \geq a_1, \end{cases} & \min(a_1, a_2) &= \begin{cases} a_1 & \text{if } a_1 \leq a_2, \\ a_2 & \text{if } a_2 \leq a_1, \end{cases} \\ \max(a_1, \dots, a_n) &= \max(\max(a_1, \dots, a_{n-1}), a_n), & \min(a_1, \dots, a_n) &= \min(\min(a_1, \dots, a_{n-1}), a_n).\end{aligned}$$

Our definitions agree with the original definitions of \max and \min for $n \leq 2$, hence our base case is already guaranteed to be true. Now suppose that our extended definitions of \max and \min compute the maximum or minimum element for a list of length $(n-1)$; we will show that this entails that they compute the correct value for a list of length n .

- Suppose $m = \max(a_1, \dots, a_n) = \max(\max(a_1, \dots, a_{n-1}), a_n) = \max(a_1, \dots, a_{n-1})$. In this case, $m \geq a_n$ by the outer \max , and by inductive assumption, $m \geq a_i$ for all $1 \leq i < n$ by the inner \max and moreover m is one of the a_i . These are the two defining properties of \max , and hence our definition computes it correctly.
- Suppose $m = a_n$ instead. In this case, $a_n \geq \max(a_1, \dots, a_{n-1})$ — but, as $\max(a_1, \dots, a_{n-1}) \geq a_i$ for $1 \leq i < n$, transitivity shows $a_n \geq a_i$ for all i in this range. Hence, a_n is the maximum element, and our definition computes \max correctly.

The proof of correctness for \min proceeds identically, with the inequalities reversed.

2.4. **Problem 5.3.22.** Show that the set S defined by $1 \in S$ and $s + t \in S$ whenever $s \in S$ and $t \in S$ is the set of positive integers.

We show both containments. First, $S \subseteq \mathbb{Z}_{\geq 1}$, as $\mathbb{Z}_{\geq 1}$ is closed under taking binary sums and $1 \in \mathbb{Z}_{\geq 1}$. The other containment is more delicate: we will inductively show that $n \in \mathbb{Z}_{\geq 1}$ is an element of S . The base case is $n = 1$, but the definition of S is that it has element 1 as an element, so we are immediately finished. Now suppose that S contains the element $(n-1)$ for some $n-1 \geq 1$; then as S also contains 1 and is closed under sums, it contains $(n-1) + 1 = n$. By induction, $\mathbb{Z}_{\geq 1} \subseteq S$, and hence $S = \mathbb{Z}_{\geq 1}$.

2.5. **Problem 5.3.36.** Let the reversal of the string w be denoted by w^R . Give a recursive definition of w^R , and use structural induction to prove it correct.

Our recursive definition is:

$$\begin{aligned}“w_0”^R &= “w_0”, \\ w^R &= (uv)^R = v^R u^R,\end{aligned}$$

where juxtaposition denotes string concatenation.

We prove that this definition is correct by strong induction on the length. In the base case of length 1 strings, our definition takes no action; this is correct, as a length 1 string has no reorderings (and so being read backward is the same as being read forward). In general, suppose we have a length n string w , which we divide into a length k initial substring u and a length $(n-k)$ final substring v : $w = uv$. By definition, we have $w^R = (uv)^R = v^R u^R$. We can then consider the i th character in w^R :

$$(w^R)_i = \begin{cases} (v^R)_i = v_{n-k-i} = w_{(k)+(n-k-i)} = w_{n-i} & \text{when } 0 \leq i < n-k, \\ (u^R)_i = u_{k-i} = w_{(n-k)+(k-i)} = w_{n-i} & \text{when } n-k \leq i < n, \end{cases}$$

as desired.

2.6. **Problem 5.3.38.** Give a recursive definition of the set of bit strings that are palindromes.

The empty string is a palindrome. All strings with one character are palindromes. All strings of the form cwc are palindromes, where c is a character and w is a palindrome.

2.7. **Problem 5.3.44.** Use structural induction to show that $\ell(T)$, the number of leaves of a full binary tree T , is 1 more than $i(T)$, the number of internal vertices of T .

The base binary tree has a single node: its root. As it has no descendants, it is thought of as a leaf, giving

$$\# \text{ of leaves} - \# \text{ of internal nodes} = 1.$$

More generally, suppose that this is true for a given pair of binary trees T_1 and T_2 , and form a new tree T from these by attaching them to a new root node. This preserves all the old leaf nodes and adds one new internal node (the new root), giving:

$$\begin{aligned} (\# \text{ of leaves of } T) - (\# \text{ of I.N.s of } T) \\ = (\# \text{ of leaves of } T_1 + \# \text{ of leaves of } T_2) - (\# \text{ of I.N.s of } T_1 + \# \text{ of I.N.s of } T_2 + 1) \\ = (\# \text{ of leaves of } T_1 - \# \text{ of I.N.s of } T_1) + (\# \text{ of leaves of } T_2 - \# \text{ of I.N.s of } T_2) - 1 \\ = 1 + 1 - 1 = 1, \end{aligned}$$

using the inductive assumptions. By structural induction, this demonstrates the equality for all full binary trees.