Math 25a Practice Final #1 Solutions

Problem 1. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

Solution. The direct sum decomposition $V = U \oplus W$ means that every vector $v \in V$ can be written uniquely as $v = u + w$ for $u \in U$ and $w \in W$. Expanding u and w in their relevant bases shows that the union of the two bases span. To see that the two bases are linearly independent, consider a putative dependence. It cannot involve vectors only from u_* or w_*, since this would restrict to give a linear dependence among two sets which are known to be linearly independent. Consider v the linear combination of just the u_* vectors, and consider v' the linear combination of just the w_* vectors. This gives a nonunique expression of the zero vector: $0 = v + v'$ and $0 = 0 + 0$. (ECP)

Problem 2. Suppose V is a K-vector-space which we do *not* assume to be finite-dimensional, and let $f\colon V \to V$ be a linear map. Show that if $\ker f$ and $\operatorname{im} f$ are finite-dimensional then so is V. (**Warning:** before using the Fundamental Theorem, think carefully about what assumptions it uses.)

Solution. Assuming $\operatorname{im} f$ is finite dimensional, pick a basis $v_1, \ldots, v_n$ and preimages $v_1', \ldots, v_n' \in V$. We claim that $\operatorname{span}\{v_1', \ldots, v_n'\} + \ker f$ is all of V (in fact, it is a direct sum), from which it follows that V is finite dimensional (since $\dim \ker f < \infty$). To see spanning, take $v \in V$ and apply f: $f(v) = k_1 v_1 + \ldots + k_n v_n$ in the chosen basis for $\operatorname{im} f$. Now consider $v' = k_1 v_1' + \ldots + k_n v_n'$. Because this has the same image as v under f, we must have $f(v - v') = 0$, or $u := v - v' \in \ker f$. This decomposes v as $v = v' + u$. (ECP)

Problem 3. Let J_n be the $n \times n$ matrix with all entries equal to one. Calculate the eigenvalues of J_n and the multiplicities of their eigenspaces.

Solution. Let v be an eigenvector of J_n with coordinates $(v_1, \dots, v_n)$. Then $J_n(v)$ is the vector with all coordinates equal to $S = \sum v_i$. Plugging this into the equation $J_n(v) = \lambda v$, we find $S = \lambda v_i$ for every i.

Adding this up for all i, we find $nS = \lambda S$. Now we have two cases. If $S \neq 0$, then we find $\lambda = n$. Furthermore, all the v_i are equal to $\frac{S}{n}$, so it follows that all the v_i are equal. Therefore, the eigenspace has multiplicity 1, being spanned by the all-ones vector.

In the other case, we have $S = 0$ and therefore since $S = \lambda v_i$ for every i, we must have $\lambda = 0$. In this case, we can check that by definition of the all-ones matrix, any vector with coordinates summing to 0 will be an eigenvector of J_n with eigenvalue 0. (RP)

Problem 4. Let V be an inner product space and suppose $\pi \colon V \to V$ is a projection operator, so that $\pi \circ \pi = \pi$. Then $V = \ker \pi \oplus \operatorname{im} \pi$. Prove that π is self adjoint if and only if $\ker \pi$ and $\operatorname{im} \pi$ are orthogonal complements.

Solution. It is always true that $\ker \pi$ and $\operatorname{im} \pi$ are complements. If they are additionally orthogonal, then we have

$$\langle \pi v, v' \rangle = \langle \pi(u + w), u' + w' \rangle = \langle \pi w, w' \rangle = \langle w, \pi w' \rangle = \langle u + w, \pi(u' + w') \rangle.$$

In the other direction, if π is self-adjoint, then $\langle u, w \rangle = \langle u, \pi v \rangle = \langle \pi u, v \rangle = \langle 0, v \rangle = 0$. (ECP)

Problem 5. Let V be an inner product space with $\dim V > 0$.

1. Suppose that V is *real*. Show that the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V, V)$, the vector space of linear maps $V \to V$.

2. Suppose instead that V is *complex*. Show that the set of self-adjoint operators on V is *not* a subspace of $\mathcal{L}(V, V)$.

Solution.

1. If $f = f^*$ and $g = g^*$, then $(f + g)^* = f^* + g^* = f + g$ and $(kf)^* = kf^*$.

2. It's not even closed under scalar multiplication. A self-adjoint operator can be expressed as a diagonal real matrix in some basis. Multiplying a nonzero such matrix by the complex scalar i gives a diagonal purely imaginary matrix, whose eigenvalues are now visibly non-real. (Alternatively, $(if)^* = \bar{i} f^* = -if^*$.) (ECP)

Problem 6.

1. State the complex spectral theorem.

2. Prove the complex spectral theorem. (You may assume that a complex operator on a finite-dimensional space admits an upper-triangular presentation in *some* basis.)

Solution.

1. For V a finite-dimensional complex inner product space and $f \colon V \to V$ an operator, f is normal if and only if f orthonormally diagonalizable.

2. We have two directions to prove. In an orthonormal basis, the conjugate transpose presents taking adjoints, and the conjugate transpose of a diagonal complex matrix agrees with just the complex conjugate. Direct calculation shows that f and f^* commute in this context. On the other hand, suppose that f is normal and select an upper triangular presentation for f. Applying Gram-Schmidt to the basis ensures that the basis is orthonormal with modifying the upper-triangularity of the presentation of f. Now consider $\|fe_1\|^2$ and $\|f^*e_1\|^2$, which are presented in terms of the matrix coefficients of f as

$$\|fe_1\|^2 = |a_{11}|^2, \qquad \|f^*e_1\|^2 = \sum_{j=1}^{n} |a_{1j}|^2.$$

Since norms are nonpositive, this forces $a_{1j} = 0$ for $j > 1$. Continuing down the rows of f in this manner shows that f and f^* are, in fact, diagonal matrices. (ECP)

Problem 7. 1. Suppose V is a complex inner product space. Prove that every normal operator on V has a square root.

2. Is the square root of a normal operator necessarily normal? Prove or give a counterexample.

Solution. 1. Put a normal operator f into orthonormal diagonal form, and let g denote the component-wise square root (in the same basis). Since f is diagonal, so is g, and direct calculation shows $gg = f$.

2. Not necessarily. The positive operator 0 has a nontrivial non-normal square root:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(ECP)

Problem 8. Let A be a complex $n \times n$ matrix that has only one eigenvalue, $\lambda_1 = 4$. Prove that if the matrix S is such that $A = SJS^{-1}$, and J is in Jordan Normal Form, then the columns of S must be generalized eigenvectors of A.

Solution. Since λ_1 is the unique eigenvalue, *every* vector must be a generalized eigenvector for λ_1. This includes all the columns of S.

Less cheekily, the generalized eigenspaces associated to an operator are independent of change of basis. The columns of a matrix in Jordan normal form explicitly say which eigenspace the Jordan basis vectors belong to, and S encodes a change of basis from some fixed basis to a Jordan basis of generalized eigenvectors. (ECP)

Problem 9. Suppose V is a complex vector space of dimension $n < \infty$, and let $f\colon V \to V$ be an invertible linear operator. For p the characteristic polynomial of f and q the characteristic polynomial of f^{-1}, demonstrate the relation

$$q(z) = \frac{1}{p(0)} \cdot z^n \cdot p\left(\frac{1}{z}\right).$$

Solution. Every $v \in V$ which is a generalized eigenvector of weight $\lambda \neq 0$ for f also serves as a generalized eigenvector of weight λ^{-1} for f^{-1}, and hence $G(\lambda, f) = G(\lambda^{-1}, f^{-1})$.[1] We thus have

$$q(z) = \prod_{\lambda} (z - \lambda^{-1})^{\dim G(\lambda^{-1}, f^{-1})} = z^{-n} \prod_{\lambda} \lambda^{-\dim G(\lambda^{-1}, f^{-1})} (\lambda - z^{-1})^{\dim G(\lambda^{-1}, f^{-1})}.$$

Recognizing the constant product as the constant term $p(0)$ of p finishes the problem. (ECP)

Problem 10. Given a complex vector space W, let $W_{\mathbb{R}}$ denote its underlying real vector space. Suppose V is a real vector space, and let $V_{\mathbb{C}}$ be its complexification. Show $(V_{\mathbb{C}})_{\mathbb{R}}$ is isomorphic to $V \oplus V$.

Solution. Let $v_1, \ldots, v_n$ be a basis of V.

First, we will show that the set $S = \{v_1, \ldots, v_n, i \cdot v_1, \ldots, i \cdot v_n\}$ is a basis of $(V_{\mathbb{C}})_{\mathbb{R}}$.

By definition, any vector in $(V_{\mathbb{C}})_{\mathbb{R}}$ can be written in the form $u + iv$ for $u, v \in V$. Since $\{v_j\}$ is a basis, there exists real scalars a_j, b_j such that $u = \sum_{j=1}^n a_j v_j$, $v = \sum_{j=1}^n b_j v_j$. It follows that $u + iv = \sum_{j=1}^n a_j \cdot v_j + b_j \cdot (i \cdot v_j)$, so S spans $(V_{\mathbb{C}})_{\mathbb{R}}$.

Now assume there exists real scalars a_j, b_j such that $\sum_{j=1}^n a_j \cdot v_j + b_j \cdot (i \cdot v_j) = 0$. Looking at the real and imaginary parts, we require $\sum a_j v_j = 0$ and $\sum b_j v_j = 0$. By linear independence of the v_j, all the a_j, b_j must equal 0 and therefore S is linearly independent as well.

Now define a map $T : (V_{\mathbb{C}})_{\mathbb{R}} \to V \oplus V$ satisfying $T(v_j) = (v_j, 0)$ and $T(i \cdot v_j) = (0, v_j)$. Since the v_j are a basis of V, the pairs $(v_j, 0)$ and $(0, v_j)$ form a basis of $V \oplus V$. Therefore, we have immediately that T is surjective.

If $T(u + iv) = 0$, then again write out $u + iv = \sum_{j=1}^n a_j v_j + b_j (i \cdot v_j)$. By linearity of T, the left-hand side is equal to $\sum_{j=1}^n a_j T(v_j) + b_j T(i \cdot v_j) = (\sum_{j=1}^n a_j v_j, \sum_{j=1}^n b_j v_j) \in V \oplus V$. By linear independence of the v_j again, we have that this is equal to 0 iff $a_j, b_j = 0$ and so $u + iv = 0$. Therefore, T must be injective as well and therefore is an isomorphism. (RP)

Problem 11. Suppose V is an inner product space and that $f, g \colon V \to V$ are orthogonal projections. Show $\operatorname{tr}(fg) \geq 0$.

Solution. Let $u_1, \ldots, u_m$ form an orthonormal spanning set for $\operatorname{im} f$ and let $w_1, \ldots, w_n$ form an orthonormal spanning set for $\operatorname{im} g$. This expresses the two operators as

$$f(v) = \sum_{i=1}^m \langle v, u_i \rangle u_i, \qquad g(v) = \sum_{j=1}^n \langle v, w_j \rangle w_j.$$

[1] This doesn't seem trivial to prove. The snappiest I can think of is to break f up into block diagonal form with upper triangular blocks (i.e., the "canonical decomposition"), then notice (1) that inverting a block diagonal matrix means inverting each individual block and (2) that inverting an upper triangular matrix gives an upper triangular result with reciprocals on the diagonal, so this new inverted matrix is also canonically decomposed.

The composition is then

$$\begin{aligned}(fg)(v) &= \sum_{i=1}^{m} \left\langle \sum_{j=1}^{n} \langle v, w_j\rangle w_j, u_i \right\rangle u_i \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} \langle v, w_j\rangle \langle w_j, u_i\rangle u_i.\end{aligned}$$

Extending $\{u_i\}$ to a basis of V, the trace is then computed by the sum

$$\begin{aligned}\operatorname{tr}(fg) &= \sum_{k} \left\langle \sum_{i=1}^{m} \sum_{j=1}^{n} \langle u_k, w_j\rangle \langle w_j, u_i\rangle u_i, u_k \right\rangle \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} \langle u_i, w_j\rangle \langle w_j, u_i\rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} |\langle u_i, w_j\rangle|^2.\end{aligned} \qquad \text{(ECP)}$$

Problem 12. Suppose A is a block upper-triangular matrix

$$A = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

where each A_j along the diagonal is a square matrix. Prove that

$$\det A = (\det A_1) \cdots (\det A_m)$$

Solution. Write n_j for the sum of the sizes of the matrices $A_1, \ldots, A_j$. The main claim is that in the permutation-sum formula expressing the determinant of A, if any of the elements $N \in (n_j, n_{j+1}]$ is sent below this range (i.e., potentially involving a term from the off-diagonal part of the upper-triangular matrix), then some earlier $M < N$ must be sent above its range (i.e., into the zero part). Namely, let $\sigma(N)$ be the image of an element $N \in (n_j, n_{j+1}]$ with $\sigma(N) \leq n_j$. Let k be the smallest index with $n_k \geq \sigma(N)$, i.e., the vertical area of the matrix where $\sigma(N)$ wound up. Since rows are not allowed to be repeated in a permutation-sum, the previous $[1, n_k]$ elements would be required to fit into $[1, n_k] \setminus \{\sigma(N)\}$ positions in order to stay in the populated rows. This is not possible: there must be an $M \leq n_k$ with $\sigma(M) > n_k$.

This shows that the only nonvanishing terms in the permutation-sum are those where the entries of A are pulled from the diagonal blocks. Factoring the leftovers appropriately (i.e., into blocks) yields the above product formula. (ECP)

Math 25a Practice Final #2 Solutions

Problem 1. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim(\operatorname{Span}\{v_1 + w, \ldots, v_m + w\}) \geq m - 1.$$

Solution. In the span, we can find the differences $(v_j+w)-(v_m+w) = v_j - v_m$. As $1 \leq j < m$ ranges, this produces $m-1$ vectors. Suppose there were a dependence

$$k_1(v_1 - v_m) + \cdots + k_{m-1}(v_{m-1} - v_m) = 0,$$

then we would have a dependence

$$k_1 v_1 + \cdots + k_{m-1} v_{m-1} - (k_1 + \cdots + k_{m-1}) v_m = 0.$$

If $k_1 + \cdots + k_{m-1}$ is nonzero, we are at a contradiction: this original set of vectors does not admit nontrivial linear dependencies. On the other hand, if it is zero, the remaining terms are also known to form a linearly independent set, so they must all be zero themselves. (ECP)

Problem 2. Suppose two linear functionals $\varphi_1, \varphi_2 \colon V \to K$ have the same kernel. Show that there exists a constant c with $\varphi_1 = c\varphi_2$.

Solution. Select a preimage v_1 of 1 along φ_1: $\varphi_1(v_1) = 1$. We claim first that $V = \operatorname{span}\{v_1\} \oplus \ker \varphi_1$ (and this is a special case of Problem #2 from the first practice exam). Since $\ker \varphi_1 = \ker \varphi_2$ and $v_1 \notin \ker \varphi_1$, it must be the case that $\varphi_2(v_1) = c^{-1}$ gives a nonzero value c. This gives comparable definitions of φ_1 and φ_2: they both send $\ker \varphi_1 = \ker \varphi_2$ to zero, and they send the direct sum complement $\operatorname{span}\{v_1\}$ to 1 and c^{-1} respectively. Since these definitions are c–multiples of one another, we have $\varphi_1 = c\varphi_2$. (ECP)

Problem 3. Suppose $f \colon \mathbb{C}^3 \to \mathbb{C}^3$ is linear and that all the eigenvalues of f are 0. Prove that $f \circ f \circ f = 0$.

Solution. Place f into upper-triangular form. Because all eigenvalues of f are zero, all the diagonal entries of this form are zero too. Manually computing the cube of such a matrix reveals it to be the zero matrix. (ECP)

Problem 4. Suppose $v_1, \dots, v_m$ is a linearly independent list in V. Show that there exists $w \in V$ such that $\langle w, v_j \rangle > 0$ for every v_j in the list.

Solution. Let $U = \operatorname{span}\{v_1, \dots, v_m\}$ be the subspace under consideration, and let $e_1, \dots, e_m \in U$ be an orthonormal list spanning it. The matrix M encoding the change of basis from v_* to e_* records their mutual inner products

$$M = \begin{pmatrix} \langle v_1, e_1 \rangle & \cdots & \langle v_m, e_1 \rangle \\ \vdots & & \vdots \\ \langle v_1, e_m \rangle & \cdots & \langle v_m, e_m \rangle \end{pmatrix},$$

in the sense that M applied to the j^{th} standard basis vector reveals the linear combination coefficients expressing v_j as a linear combination of the e_* vectors. Because this is a change-of-basis matrix, it is of full rank, and hence so is its conjugate-transpose M^*. For any positive vector y (e.g., $y = (1, 1, \dots, 1)$), we can thus solve the equation $M^* x = y$. We claim that the linear combination w of the e_* vectors encoded by x solves the problem. Namely:

$$\begin{aligned} \langle w, v_j \rangle &= \langle x_1 e_1 + \cdots + x_m e_m, v_j \rangle \\ &= x_1 \langle e_1, v_j \rangle + \cdots + x_m \langle e_m, v_j \rangle \\ &= x_1 \overline{\langle v_j, e_1 \rangle} + \cdots + x_m \overline{\langle v_j, e_m \rangle} = y_j = 1. \end{aligned} \qquad \text{(ECP)}$$

Problem 5. Let $f\colon V \to V$ be a linear operator and let p be a polynomial. Show that $\ker f \subseteq \ker p(f)$ if and only if p has constant term equal to zero.

Solution. Suppose first that p has no constant term, and let $v \in \ker f$ be a kernel vector. Then

$$p(f)(v) = (a_1 f + a_2 f^2 + \cdots + a_n f^n)(v) = 0.$$

Alternatively, suppose that p does have a constant term, and let $v \in \ker f$ be any *nonzero* kernel vector. Then

$$p(f)(v) = (a_0 + a_1 f + \cdots + a_n f^n)(v) = a_0 v \neq 0. \qquad \text{(ECP)}$$

Problem 6. Let $f\colon V \to V$ be an operator on a finite-dimensional inner product space. Show that f is invertible if and only if there exists a *unique* linear isometry $g\colon V \to V$ with $f = g \circ \sqrt{f^* f}$.

Solution. Suppose first that f is invertible. The decomposition $f = g \circ \sqrt{f^* f}$ forces $\sqrt{f^* f}$ to be invertible as well (so 0 cannot appear as a singular value of f). In that case, two putative decompositions $f = g \circ \sqrt{f^* f}$ and $f = g' \circ \sqrt{f^* f}$ can be rearranged to give $f \circ (\sqrt{f^* f})^{-1} = g$ and $f \circ (\sqrt{f^* f})^{-1} = g'$ respectively, so that $g = g'$. On the other hand, if f fails to be invertible, then $\operatorname{im} f$ cannot span all of V. On the orthogonal complement, we can pick any nontrivial isometry and extend by the identity on $\operatorname{im} f$ to get a total isometry $\tilde{g}\colon V \to V$ such that $f = g \circ \sqrt{f^* f} = \tilde{g} \circ g \circ \sqrt{f^* f}$ for *any* initial polar decomposition. (ECP)

Problem 7. Let V be a complex inner product space and let $f\colon V \to V$ be a linear isometry.

1. Prove that all eigenvalues of f have absolute value 1.
2. Prove that the determinant of f has absolute value 1.
3. Prove that $g\colon V \to V$ is a linear isometry if and only if for any orthonormal basis α_* of V, $g(\alpha_*)$ is also an orthonormal basis.

Solution. 1. For v an eigenvector, the isometry equation $\|v\| = \|fv\| = |\lambda|\|v\|$ forces $|\lambda| = 1$.

2. The determinant of f is the product of the eigenvalues:

$$|\det f| = \left|\prod_j \lambda_j\right| = \prod_j |\lambda_j| = 1.$$

3. Beginning with a vector v, normalize it to $\alpha_1 = v/\|v\|$ and extend to an orthonormal basis α_*. Then $g(\alpha_*)$ is again orthonormal, and in particular $\|g(\alpha_1)\| = 1$. We have therefore calculated

$$\|g(v)\| = \|g(\|v\| \cdot \alpha_1)\| = \|\|v\| \cdot g(\alpha_1)\| = \|v\|.$$

Instead suppose that g has the isometry property, and let α_* be some orthonormal basis. Because $\|\alpha_*\| = \|g(\alpha_*)\|$, we see immediately that g carries α_* to a set of unit vectors. We also learned on the homework that the norm determines the inner product, hence if g is a linear map preserving the norm, it must also preserve the inner product. It follows that $g(\alpha_*)$ is an orthonormal set. (ECP)

Problem 8. Suppose $f\colon \mathbb{C}^6 \to \mathbb{C}^6$ is a linear operator such that…

- …its minimal polynomial is $(z-1)(z-2)^2$.
- …its characteristic polynomial is $(z-1)^2(z-2)^4$.

Describe the possible presentations of f as a matrix in a Jordan basis.

Solution. The characteristic polynomial alone determines the following forms for the matrix:

$$\begin{pmatrix} 1 & ? & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & ? & 0 & 0 \\ 0 & 0 & 0 & 2 & ? & 0 \\ 0 & 0 & 0 & 0 & 2 & ? \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The minimal polynomial carries two pieces of information: the exponents simultaneously determine the size of the largest Jordan block *and* assert the existence of at least one Jordan block of exactly that size. This reduces the possibilities to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & ? \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

where the remaining "?" is either a 0 or a 1. (ECP)

Problem 9. Suppose V is a finite-dimensional complex vector space with an operator f, and let $V_1, \ldots, V_m$ be nonzero invariant subspaces decomposing V as

$$V = V_1 \oplus \cdots \oplus V_m.$$

For each j, let p_j denote the characteristic polynomial of $f_j = f|_{V_j}$ and let q_j denote the minimal polynomial of f_j.

1. Show that the characteristic polynomial of f is the product $p_1 \cdots p_m$.

2. Show that the minimal polynomial of f is the least common multiple of $q_1, \ldots, q_m$.

Solution. 1. Find basis $v_{j,*}$ for each V_j presenting f as an upper-triangular matrix. The characteristic polynomial for f (resp., f_j) is given by the product of the monic linear factors with a root at each diagonal entry of the whole matrix (resp., the part governed by f_j). These parts assemble into the whole, as claimed.

2. The least common multiple is computed to be

$$\operatorname{lcm}(z) = \prod_\lambda (z - \lambda)^{\min_j\{\dim G(\lambda, f_j)\}}.$$

This polynomial is a monic polynomial such that $\operatorname{lcm}(f)$ kills all of V: any vector in V decomposes as a sum of vectors in V_j, and on each V_j the polynomial $\operatorname{lcm}(z)$ is divisible by p_j, hence it kills v. On the other hand, suppose it were not minimal: since it has the same roots as $p(z)$, it must be the case that one of the exponents in the factorization of $p(z)$ is lower than the corresponding exponent present in $\operatorname{lcm}(z)$. Suppose that V_j determined that exponent and that λ is the relevant eigenvalue. Since $p(f)$ kills all of V, it must also kill all of V_j, and in particular all of $G(\lambda, f_j)$. This cannot be the case, as p_j was already selected to be the minimal-degree polynomial killing this subspace. (ECP)

Problem 10. Suppose V is a finite-dimensional real vector space and $f \colon V \to V$ is an operator on V satisfying $f^2 + bf + c = 0$ for some $b, c \in \mathbb{R}$. Show that f has an eigenvalue if and only if $b^2 \geq 4c$.

Solution. The condition $b^2 \geq 4c$ is exactly the condition that there exist $r, s \in \mathbb{R}$ such that $x^2 + bx + c = (x - r)(x - s)$. Since $(f - r)(f - s) = 0$, it must be the case that at least one of the two factors has nontrivial kernel, and any member of either kernel witnesses an eigenvector of f. Conversely, if f has an eigenvector v satisfying $fv = r \cdot v$ for some $r \in \mathbb{R}$, then $x - r$ must cleanly divide the minimal polynomial of f. Since $x^2 + bx + c$ is a monic polynomial killing f, it is divided by the minimal polynomial, and chaining these together shows that $x - r$ is a factor of $x^2 + bx + c$. Hence, $x^2 + bx + c = (x - r)(x - s)$ for some other $s \in \mathbb{R}$, which shows $b^2 \geq 4c$. (ECP)

Problem 11. Let V be an inner product space, and fix vectors $u, w \in V$. Define $f\colon V \to V$ by the formula

$$f(v) = \langle v, u \rangle \cdot w.$$

Calculate $\operatorname{tr} f$.

Solution. Remembering the slogan "the trace is a linearized dimension function," we seek to make f look as much like a sum of projections as possible. Begin by extending u to an orthogonal basis $(u, u_2, u_3, \ldots, u_n)$ of V. We can thus decompose w as

$$w = ku + k_2 u_2 + k_3 u_3 + \cdots + k_n u_n,$$

which gives

$$f(v) = \langle v, u \rangle \cdot (ku + k_2 u_2 + k_3 u_3 + \cdots + k_n u_n).$$

We calculate the trace by

$$\begin{aligned}
\operatorname{tr} f &= \sum_j \frac{\langle f(u_j), u_j \rangle}{\|u_j\|^2} \\
&= \sum_j \frac{\langle \langle u_j, u \rangle \cdot w, u_j \rangle}{\|u_j\|^2} \\
&= \frac{\langle \langle u, u \rangle \cdot (ku + k_2 u_2 + \cdots + k_n u_n), u \rangle}{\|u\|^2} \\
&= k\|u\|^2 = \frac{\langle w, u \rangle}{\|u\|^2} \|u\|^2 = \langle w, u \rangle.
\end{aligned}$$

(ECP)

Problem 12. Let V be a vector space with basis $v_0, \ldots, v_n$ and let $a_0, \ldots, a_n$ be scalars. Define a linear map f on V by its action on a basis $f(v_i) = v_{i+1}$ and $f(v_n) = a_0 v_0 + a_1 v_1 + \ldots + a_n v_n$. Determine the matrix of f with respect to the given basis, and the characteristic polynomial of f.

Solution. The matrix has 1s on the sub-diagonal, the column vector $(a_0, \ldots, a_n)$ in the final column, and 0s otherwise. Beginning with the vector v_0, the list $(v_0, fv_0, f^2 v_0, \ldots, f^n v_0)$ is linearly independent, forcing the degree of the minimal polynomial up to at least $n + 1$. Hence, the minimal and characteristic polynomials agree. The polynomial

$$p(x) = x^{n+1} - a_n x^n - a_{n-1} x^{n-1} - \cdots - a_1 x - a_0$$

kills v_0 by direct computation. Additionally, the other basis vectors $v_j = f^j v_0$ are also killed by this polynomial:

$$p(f)(v_j) = p(f)(f^j v_0) = f^j(p(f)(v_0)) = f^j(0) = 0.$$

It follows that p is the minimal polynomial, since it is monic, $p(f)$ kills V, and p is of the right degree. Since it is the minimal polynomial, we have already concluded that it is also automatically the characteristic polynomial. (ECP)

Math 25a Final Exam Solutions

Problem 1. Suppose we have a K-vector space V of dimension at least 3, as well as distinct vectors $u, v \in V$. Prove that $\{u, v\}$ is linearly independent if and only if for some $w \notin \operatorname{span}\{u, v\}$ the set $\{u + v, u + w, v + w\}$ is linearly independent.

Solution. We start by showing that $\{u+v, u+w, v+w\}$ has the same span as $\{u, v, w\}$. One direction is done for us, since every vector in the first list is a linear combination of vectors in the second list. In reverse, note that we can construct the difference $(u+w)-(v+w) = u-v$, and hence the sums $(u + v) + (u - v) = 2u$ and $(u + v) - (u - v) = 2v$. Halving these shows we can construct u and v as linear combinations of vectors in the second list. Once we have v, we can finally form the difference $(v + w) - v = w$, and hence the two lists have the same span. Since w is chosen to be linearly independent of u and v, the first list is linearly independent if and only if the list $\{u, v\}$ is. Chaining these together, we have that the second list is linearly independent if and only if the first list is linearly independent if and only if the list $\{u, v\}$ is linearly independent. (ECP)

Problem 2. Consider the block matrix

$$M = \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right)$$

for auxiliary matrices A, B, and C. Show the inequality

$$\operatorname{rank} A + \operatorname{rank} C \leq \operatorname{rank} M \leq \operatorname{rank} A + \operatorname{rank} B + \operatorname{rank} C.$$

Solution. Block matrices encode linear transformations between vectors spaces with direct sum decompositions. Suppose M represents a map $V \to W$, decompose V and W as $V = V_1 \oplus V_2$ and $W = W_1 \oplus W_2$, and denote the induced maps by

$$A\colon V_1 \to W_1, \qquad B\colon V_1 \to W_2, \qquad C\colon V_2 \to W_2.$$

The second inequality is easier to establish. We have $\operatorname{im} M \subseteq \operatorname{im}(A \oplus 0) + \operatorname{im}(B \oplus 0) + \operatorname{im}(0 \oplus C)$, which induces an inequality on dimensions. The sum of subspaces in turn has its dimension bounded by the sum of the dimensions:

$$\dim \operatorname{im} M \leq \dim(\operatorname{im}(A \oplus 0) + \operatorname{im}(B \oplus 0) + \operatorname{im}(0 \oplus C)) \leq \dim \operatorname{im} A + \dim \operatorname{im} B + \dim \operatorname{im} C.$$

To establish the second inequality, consider $n = \operatorname{rank} A$ many input vectors $v_1, \dots, v_n \in V$ such that $A(v_*)$ are linearly independent, and consider $m = \operatorname{rank} C$ many input vectors $v'_1, \dots, v'_m \in V$ such that $C(v'_*)$ are linearly independent. We claim that these are still linearly independent under M. A dependence gives

$$0 = k_1 M v_1 + \cdots + k_n M v_n + k'_1 M v'_1 + \cdots + k'_m M v'_m,$$

which we can arrange into W_1 and W_2 components as

$$= \overbrace{(k_1 A v_1 + \cdots + k_n A v_n + k'_1 B v'_1 + \cdots + k'_m B v'_m)}^{W_1 \text{ component}} + \overbrace{(k'_1 C v'_1 + \cdots + k'_m C v'_m)}^{W_2 \text{ component}}.$$

Since the decomposition $W = W_1 \oplus W_2$ is direct and $Cv'_1, \dots, Cv'_m$ are linearly independent, it must be the case that $k'_1 = \cdots = k'_m = 0$. Putting that information back into the W_1 component, the coefficients $k_1, \dots, k_n$ must also be zero. Hence, we have a linearly independent set of length $\operatorname{rank} A + \operatorname{rank} C$ inside of $\operatorname{im} M$, bounding its dimension from below. (ECP)

Problem 3. Recall that for an operator $f \colon V \to V$ with kernel $U \subset V$, we defined an induced operator $f/U \colon V/U \to V/U$ sending $v + U$ to $f(v) + U$.

Assume f is diagonalizable and V is finite-dimensional. Show that λ is an eigenvalue of f/U if and only if $\lambda \neq 0$ and λ is an eigenvalue of f.

Solution. Assume that λ is a nonzero eigenvalue of f with eigenvector v. Then by definition of f/U, we have $f/U(v + U) = f(v) + U = \lambda v + U = \lambda(v + U)$, so $v + U$ is an eigenvector of f/U with eigenvalue λ.

Let k be the dimension of the kernel U and let n be the dimension of V. Since f is diagonalizable, it exhibits a basis of eigenvectors $v_1, \dots, v_n$ where we assume $v_1, \dots, v_k$ span U.

Then the vectors $v_{k+1} + U, \dots, v_n + U$ by the above reasoning are eigenvectors in V/U. Furthermore, no linear combination of $v_{k+1}, \dots, v_n$ can be in U since they are linearly independent and none of the $v_{k+1}, \dots, v_n$ are in U themselves.

However, the dimension of the quotient is $n - k$. Therefore, the $v_{k+1} + U, \dots, v_n + U$ form a basis of eigenvectors of f/U. Any eigenvalue of f/U therefore must be an eigenvalue of one of the $v_i + U$. By definition, this means that the eigenvalue is both nonzero and also an eigenvalue of f as desired. (RP)

Problem 4.

1. Show that $\langle A, B \rangle := \operatorname{tr}(\mathrm{AB})$ is an inner product on the space of $n \times n$ real symmetric matrices (or, equivalently, on the space of real self-adjoint operators on $\mathbb{R}^n$).

2. What is the dimension of the subspace W of such matrices additionally satisfying $\operatorname{tr}(\mathrm{A}) = 0$? What is the dimension of $W^\perp$ relative to the above inner product?

Solution. 1. Inner products are positive-definite skew-symmetric pairings. To check skew-symmetry, we have

$$\langle A, B\rangle = \mathrm{tr}(AB) = \mathrm{tr}(BA) = \langle B, A\rangle\,,$$

using the cyclic permutation property of trace. To see positive-definiteness, we know that trace is basis-invariant and that a real symmetric matrix A admits a diagonal presentation D, and hence

$$\langle A, A\rangle = \mathrm{tr}(AA) = \mathrm{tr}(DD) = \mathrm{tr}\begin{pmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{pmatrix} = \sum_j \lambda_j^2.$$

This sum of nonnegative numbers is nonnegative, and it is positive exactly when A is not the zero matrix.

2. Write V for the space of such matrices. Since trace is a linear map $\mathrm{tr}\colon V \to \mathbb{R}$, we have $\dim V = \dim \operatorname{im} \mathrm{tr} + \dim \ker \mathrm{tr}$. There are two options for $\dim \operatorname{im} \mathrm{tr}$, since $\operatorname{im} \mathrm{tr} \subseteq \mathbb{R}$, and it must be the case that $\dim \operatorname{im} \mathrm{tr} = 1$ since there exist matrices with nonzero trace (i.e., trace is not just the zero functions). To determine $\dim V$, note that a symmetric real matrix is determined by its upper-triangular portion (on which there is no further restriction), so that $\dim V = \frac{n(n+1)}{2}$. It follows that $W = \ker \mathrm{tr}$ has dimension $\frac{n(n+1)}{2} - 1$. Finally, the perpendicular to any subspace is a complement, hence $\dim W + \dim W^\perp = \dim V$, so that $\dim W^\perp = \dim V - \dim W = \frac{n(n+1)}{2} - \left(\frac{n(n+1)}{2} - 1\right) = 1$. (ECP)

Problem 5. 1. Suppose that $u, v \in V$ are vectors with $\langle u, v\rangle$ real. Show that $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2$.

2. Suppose u and v additionally satisfy $\|u\|^2 = \|v\|^2 = \langle u, v\rangle = c \in \mathbb{R}$. Prove that $u = v$.

Solution. 1. Just compute:

$$\langle u+v, u-v\rangle = \langle u, u\rangle - \langle v, v\rangle + \langle v, u\rangle - \langle u, v\rangle = \|u\|^2 - \|v\|^2 + \langle v, u\rangle - \overline{\langle v, u\rangle}.$$

The final two terms cancel in the case that $\langle u, v\rangle$ is real.

2. Showing $u = v$ is the same as showing $u - v = 0$, which is the same as showing $\|u - v\|^2 = 0$. Again, we compute:

$$\|u-v\|^2 = \langle u-v, u-v\rangle = \langle u, u\rangle + \langle v, v\rangle - \langle u, v\rangle - \langle v, u\rangle = c + c - c - c = 0.$$ (ECP)

Problem 6. Let V and W be finite dimensional real inner product spaces.

1. Define what it means for an operator $f\colon V \to V$ to be positive.

2. For any linear map $f\colon V \to W$, show that $f^*f\colon V \to V$ and $ff^*\colon W \to W$ are positive operators on V and W respectively.

3. Polar decomposition states that any operator $f\colon V \to V$ factors as a positive operator followed by a linear isometry. Show that this positive operator *must* be $\sqrt{f^* f}$.

Solution. 1. The operator f is positive when f is self-adjoint and $\langle fv, v\rangle \geq 0$ for all $v \in V$.

2. To see $f^* f$ is self-adjoint, calculate $(f^* f)^* = f^* f^{**} = f^* f$. To see the inequality, calculate $\langle f^* fv, v\rangle = \langle fv, fv\rangle = \|fv\|^2 \geq 0$. The other case is similar.

3. Let $f = gp$ be some putative polar decomposition. We calculate $f^* = (gp)^* = p^* g^* = p^* g^{-1}$ using the fact that g is an isometry in the last step. Composing this with f gives $f^* f = (p^* g^{-1})(gp) = p^* p$. Both the right- and left-hand sides are positive operators, positive operators have unique positive square roots, and p is a square root of $p^* p$, so $\sqrt{f^* f} = \sqrt{p^* p} = p$. (ECP)

Problem 7. Let V be a complex inner product space and let $f\colon V \to V$ be an operator satisfying $f^* = -f$. We call such an operator *skew-self-adjoint*.

1. Show that the eigenvalues of f are purely imaginary.

2. Show further that V decomposes as the orthogonal direct sum

$$V = \bigoplus_{\lambda \in \mathbb{R}} E_{i\lambda}(f).$$

Solution. 1. Suppose v is an eigenvector with eigenvalue λ. Then we calculate

$$\lambda\|v\|^2 = \langle fv, v\rangle = \langle v, f^* v\rangle = \langle v, -fv\rangle = \langle v, -\lambda v\rangle = -\overline{\lambda}\|v\|^2.$$

This is only satisfiable for $\lambda = a + bi$ if $a = 0$.

2. The skew-self-adjointness condition guarantees normality:

$$ff^* = f(-f) = -ff = (-f)f = f^* f.$$

As consequence, the complex spectral theorem states that V admits an orthonormal basis of eigenvectors for f. Since the eigenvalues of f are all purely imaginary, it follows immediately that V decomposes (using any of these orthonormal bases) into an orthogonal direct sum of eigenspaces of purely imaginary weight. (ECP)

Problem 8. Consider the matrix M described by

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculate a square root of M.

Solution. Since $N = I - M$ is nilpotent of order 2, we must compute the Maclaurin series for $\sqrt{1+x}$ out to second order:

$$T_2(\sqrt{1+x}) = 1 + \frac{1}{2}x - \frac{1}{4}x^2.$$

By calculating

$$N^2 = \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we use this formula to compute

$$\sqrt{M} = \sqrt{I+N} = I + \frac{1}{2}N - \frac{1}{8}N^2 = \begin{pmatrix} 1 & 1 & -3/4 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

You can prove that this answer works by making a generic argument about Taylor series and polynomial multiplication, or you can check outright that its square returns M again. (ECP)

Problem 9. Suppose $f\colon V \to V$ is an operator on a finite-dimensional complex vector space, and let $v \in V$ be a fixed vector.

1. Show that there exists a *unique* monic polynomial p of smallest degree such that $p(f)(v) = 0$.

2. Prove that p divides the minimal polynomial of f.

Solution.

1. Consider the set of monic polynomials p satisfying this vanishing condition $p(f)(v) = 0$. This set is nonempty: the list $(v, fv, \ldots, f^{\dim V} v)$ is of length larger than the dimension of V, hence has a linear dependence, from which we can extract such a polynomial. We can select some polynomial of smallest degree from this set. To see that this choice is unique, if there were two distinct polynomials of smallest degree with this condition, then their difference (divided by its leading nonzero coefficient) would be another monic polynomial of smaller degree satisfying the vanishing condition—a contradiction.

2. Polynomial division guarantees the existence of polynomials q and r such that minpoly $= p \cdot q + r$, where $\deg r < \deg p$. Since $\mathrm{minpoly}(f) = 0$ it also vanishes on v, and $p(f)(v) = 0$ by assumption, which forces $r(f)(v) = 0$ as well. This is only possible if $r = 0$ is the zero polynomial, lest the bound on its degree contradict our minimality assumption on p. The statement $r = 0$ is identical to the statement that p cleanly divides the minimal polynomial. (ECP)

Problem 10. Suppose $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear isometry. Prove that there exists a nonzero vector $v \in \mathbb{R}^3$ such that $g^2(v) = v$.

Solution. Because g is an operator on an *odd–dimensional* real vector space, it is guarateed to admit an eigenvector v of weight $\lambda \in \mathbb{R}$. The isometry condition applied to $g(v)$ gives

$$\|v\| = \|gv\| = \|\lambda v\| = |\lambda| \cdot \|v\|.$$

The only λ satisfying this equation are 1 and -1. In either case, $gg(v) = \lambda^2 v = v$ works. (ECP)

Problem 11. Suppose V is a finite-dimensional real vector space and $f\colon V \to V$ is an operator with $\operatorname{tr} f^2 < 0$. Show that f does not admit an upper-triangular presentation.

Solution. Suppose instead that f did admit an upper-triangular presentation. The product of two upper-triangular matrices behaves like component-wise multiplication on the main diagonal, and hence the diagonal of the square of an upper-triangular matrix is the square of the diagonal of the matrix. For (λ_j) the components of an upper-triangular presentation of f, we thus have $\operatorname{tr} f^2 = \sum_j \lambda_j^2$. This sum is nonnegative, demonstrating the contrapositive of the claim. (ECP)

Problem 12. Suppose V is a complex inner product space and $T\colon V \to V$ is a linear operator. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T, repeated according to multiplicity. Suppose

$$\begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{pmatrix}$$

is the matrix of T with respect to some orthonormal basis of V. Prove that

$$|\lambda_1|^2 + \ldots + |\lambda_n|^2 \le \sum_{k=1}^{n} \sum_{j=1}^{n} |a_{j,k}|^2$$

with equality when T is normal.

Solution. There is a trick from the homework that is useful here:

$$\operatorname{tr}(T^*T) = \sum_{k=1}^{n} \sum_{j=1}^{n} |a_{j,k}|^2$$

for a matrix presentation of T in an orthonormal basis as given above. Meanwhile, we also know that trace is independent of choice of basis, so this agrees with the same formula computed in any other basis. Because we are working in a complex inner product space, we know (by Schur's theorem) that we can find an orthonormal basis in which T presents as an upper-triangular matrix with its eigenvalues appearing on the main diagonal. We thus have

$$\begin{aligned} \sum_{k=1}^{n} \sum_{j=1}^{n} |a_{j,k}|^2 &= \sum_{j=1}^{n} |\lambda_j|^2 + (\text{squares of other matrix entries in the upper triangular-form}) \\ &\ge \sum_{j=1}^{n} |\lambda_j|^2. \end{aligned}$$

In the case that T is normal, the upper-triangular presentation is actually diagonal, hence the omitted term in the intermediate expression is zero, hence the inequality is an equality. (ECP)