EXPONENTIALS AND THE EULER IDENTITY

1. Dissecting the limit

In a precalculus course, the special number e is often introduced with the expression

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n,$$

or some equivalent thing without the word "limit" appearing. This appears frequently in the context of a sea of interest rate problems in an effort to motivate the study of such expressions; the number e appears as the base in the limit of compounding interest to continuously compounding interest.

Then, when discussing transcendental functions in a calculus course, a new observation about the exponential is discussed:

$$\frac{d}{dx}\exp(x)\bigg|_{x=a} = e^a.$$

This property is stated, usually without proof, as characterizing the exponential — that is, the exponential $f(x) = \exp(x)$ is the unique function f satisfying the differential equation $f'(a) = f(a)$ for all $a \in \mathbb{R}$.

Let's begin by justifying the equivalence of these two definitions. Recall that there is an inexact method to solving differential equations described as slope fields, called **Euler's method**. Euler's method moves out through the slope field using well-approximating tangent lines; for a fixed step size Δx and starting point a, we set

$$x_n = a + n\Delta x, \qquad y_n = y_{n-1} + \Delta x \cdot f'(x_{n-1}, y_{n-1}).$$

For Δx sufficiently small, we might hope that we have $y_n \approx f(x_n)$, where f solves the differential equation described by f'.[1]

Let's complicate things slightly before proceeding: fix a real number k, and consider the function $f(x) = \exp(kx)$. Then

$$\frac{d}{dx}f(x)\bigg|_{f(x)=a} = k\exp(kx)|_{f(x)=a} = ka.$$

We're going to use Euler's method to solve the associated differential equation $f'(x,y) = ky$ and $\exp(k\cdot 0) = \exp(0) = 1$. If we solve this differential equation on the interval $[0,1]$, then when $x = 1$ we'll have an approximate value for $\exp(k\cdot 1) = \exp k$, and so a formula for the exponential.

We'll set the step size to $\Delta x = 1/n$ and make n iterations to get to $x_n = 1$. We've arranged that $y_0 = 1$, and so we compute

$$y_1 = y_0 + \Delta x \cdot f'(x_0, y_0) = 1 + \frac{k}{n}.$$

Straight away, we can compute the next iteration as well:

$$\begin{aligned} y_2 &= y_1 + \Delta x \cdot f'(x_1, y_1) \\ &= \left(1+\frac{k}{n}\right) + \frac{1}{n}k\cdot\left(1+\frac{k}{n}\right) \\ &= \left(1+\frac{k}{n}\right)^2. \end{aligned}$$

[1] This requires some kind of smoothness condition on f. Our exponential function, if it exists, is going to be infinitely differentiable, and so we're safe.

We can already see a pattern in the calculation: it sure looks like $y_m = \left(1+\frac{k}{n}\right)^m$. This is indeed the case:

$$\begin{aligned} y_m &= y_{m-1} + \Delta x \cdot f'(x_{m-1}, y_{m-1}) \\ &= y_{m-1} + \frac{1}{n} \cdot k y_{m-1} \\ &= y_{m-1} \cdot \left(1+\frac{k}{n}\right) \\ &= \left(1+\frac{k}{n}\right)^m. \end{aligned}$$

Hence, $y_n = \left(1+\frac{k}{n}\right)^n$, and we get the limiting calculation $e^k = \lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n$, as desired, having begun with the differential equation definition.

2. The complex exponential

Clearly, the differential equation definition of the exponential is a powerful tool, but in fact we can push this further: let's take our method in the previous section to be the definition of the exponential. Pick a velocity k with $f'(0) = k$ and set $f(0) = 1$. We can build a slope field V with $V_a = a \cdot k$ by multiplying in a, and we define the exponential $\exp k$ to be $f(1)$, where f is a solution to this slope field, called an **integral curve**. This is a rephrasing of the above differential equations problem.

Now, let's aim high and use this to compute $\exp(i\pi)$; we'll modify the technique a little bit, so that we instead build the slope field V associated to i and follow the integral curve out π units. Recall that for a point $z = x + iy$ in the complex plane $\mathbb{C}$, we can compute $iz = i(x+iy) = (-y) + ix$. Here's a graph of the resulting slope field:

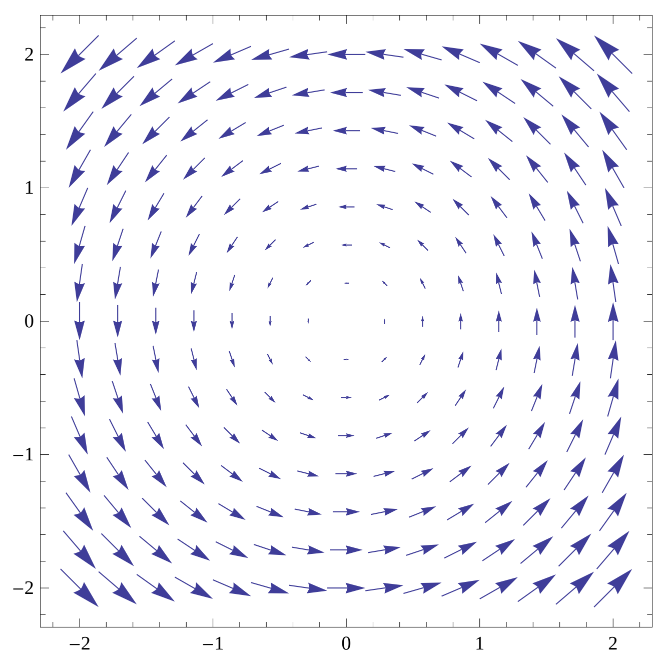

FIGURE 1. Slope field on $\mathbb{C}$ associated to i.

You can see that it sort of spirals around, and you can imagine that, starting at $z = 1+0i$ and following the arrows around for π units, you'll end up on the other end of the semicircle you've traced out, at $-1+0i$. We'll prove exactly that, assuming some knowledge of solutions to second order linear differential equations. Modeling the complex place $\mathbb{C}$ as the real plane $\mathbb{R}^2$ with the ability to multiply, we're interested in a pair of functions $x, y : \mathbb{R} \to \mathbb{C}$ so that the curve $\gamma(t) = x(t) + iy(t) = (x+iy)(t)$ in the plane is an integral curve for our vector field V, i.e., $\gamma'(t) = i\gamma(t)$.

In terms of our coordinate functions x and y, this imposes the relation $(x'(t), y'(t)) = (-y(t), x(t))$. We differentiate again to get the following important relation:

$$(x''(t), y''(t)) = (-y'(t), x'(t)) = (-x(t), -y(t)).$$

So, both x and y satisfy the differential equation $f''(t) = -f(t)$. The difference between them is that, since $\gamma(0) = 1 + 0i = (1, 0)$ and $\gamma'(0) = 0 + 1i = (0, 1)$, we have the distinct initial conditions

$$\begin{aligned} x(0) &= 1, & y(0) &= 0, \\ x'(0) &= 0, & y'(0) &= 1. \end{aligned}$$

These functions have names: cosine and sine! The value π is **defined** to be the first nonzero positive root of sine, and so we compute $\gamma(\pi) = \cos\pi + i\sin\pi = -1 + i \cdot 0 = -1$. This gives Euler's identity:

$$e^{i\pi} + 1 = 0.$$

3. Remarks

This technique is absurdly general. In any context where you can multiply and differentiate, called a **Lie group**, you can build an exponential function which sends tangent velocities to points in the Lie group. The two Lie group exponentials we've investigated in this note are

$$\mathbb{R} \cong T_1\mathrm{GL}_1(\mathbb{R}) \to \mathrm{GL}_1(\mathbb{R}) = \mathbb{R} \setminus \{0\}$$

and

$$\mathbb{C} \cong T_1\mathrm{GL}_1(\mathbb{C}) \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}.$$

Well-behavedness of the initial conditions that go into building the vector field V demonstrate that the exponential is always a continuous map, which in turn tells you why the standard real-valued exponential is always nonnegative; its image has to land in the positive connected component of $\mathrm{GL}_1(\mathbb{R})$. This also explains other exponentials you'll counter: the matrix exponential, for instance, is built in exactly the same way.