$\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3] = V' + \mathcal{L}_{12} + \mathcal{L}_{23} + \mathcal{L}_{31} - \mathcal{L}_{123} - (r-1)\cdot 1$ ‖ $[a][b] = [a+b] - [a]$

130: Classical Geometry

@MATH.HARVARD.EDU

: 321h SC, MT 2-3 pm.

: Homeworks 50%, Midterm paper 20%, final paper 30%

://math.harvard.edu/~ecpl/teaching/Spring2016/130/.

2/22, 3/4, 3/11 — 4/11, 4/22, 5/5.

illars: Axioms, coordinates, projective geom., symmetry gps.

ightedge + compass.

can: $+, -$. $1/n$ for $n \in \mathbb{N}$, a/b, $a \cdot b$. 17, 257, 65537

onstruct some regular polygons (p-gons, p prime $\Rightarrow p = 2^{2^m}+1$.)

Half but not trisect an angle. Compute areas. Congruence.

$\cancel{\mathbb{Z}/p \to \mathbb{Z}/p}$

lly do

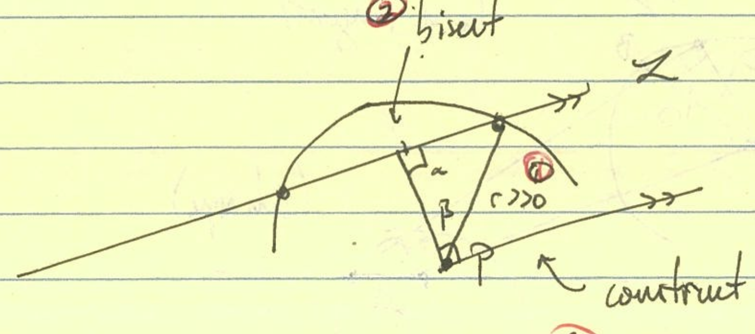
Δ.

$\cancel{\mathbb{Z}/p^2 \hookrightarrow \mathbb{Z}/p^2 \otimes_{\mathbb{Z}} \mathbb{Z}/p}$

halve a line.

Thales's Thm + Length Arithmetic

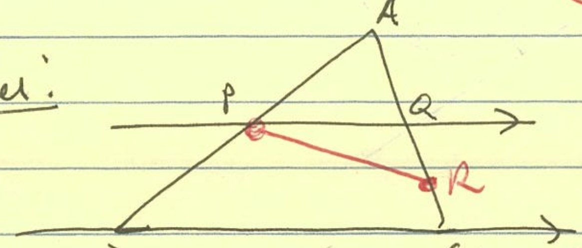
Construct a parallel to a line through a point:



Def^n of $\parallel$: all angles agree (not just vertical ones).

Parallel postulate: If $\alpha + \beta < \pi$, the lines meet.

Thales:

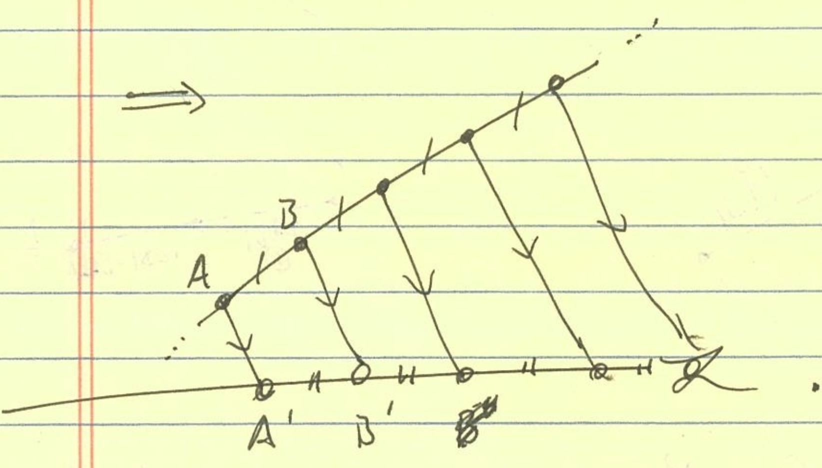


$$\frac{|AP|}{|PB|} = \frac{|AQ|}{|QC|}$$ (proportionality)

(We will prove this eventually.)

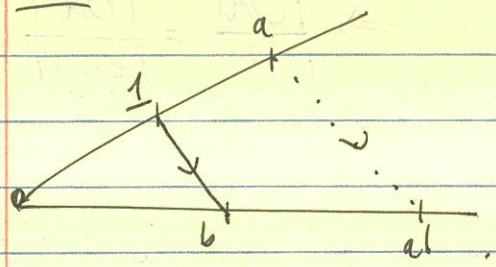
(If + only if.)

$\Longrightarrow$

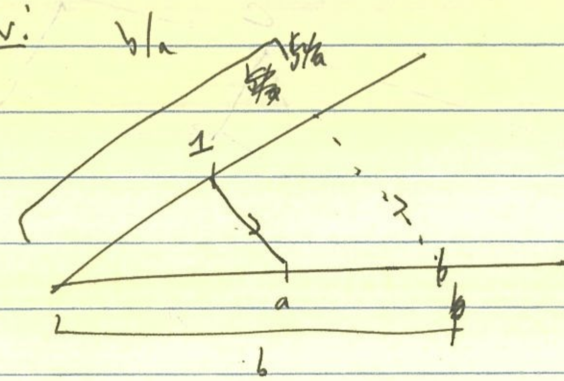


Q: How can you divide any line segment into n parts?

Mult:

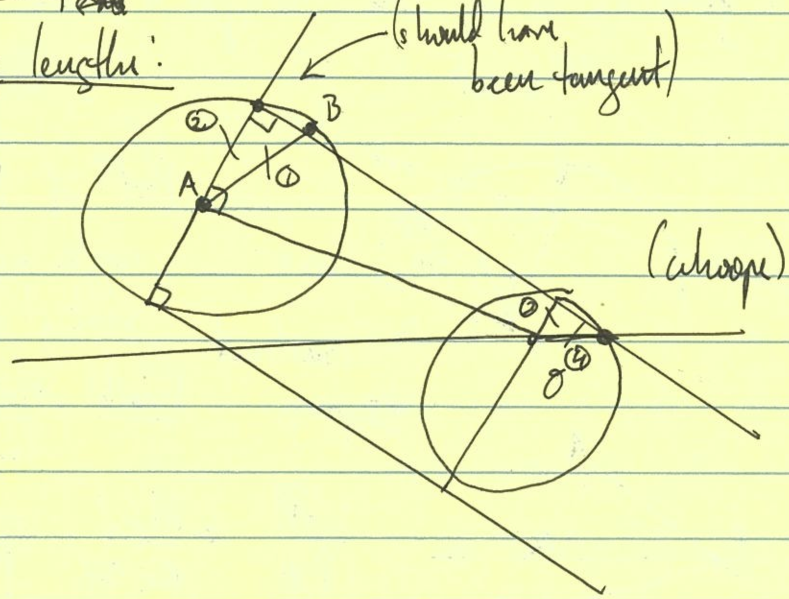


Div:

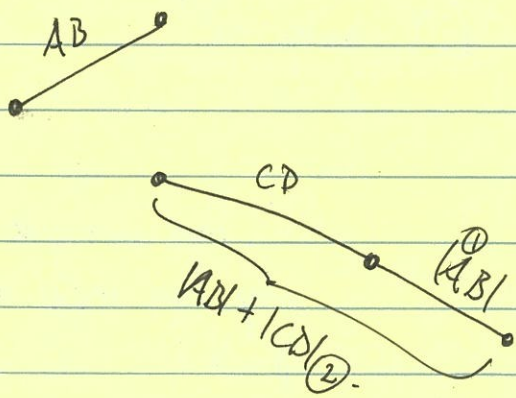


~~Addition: Form~~

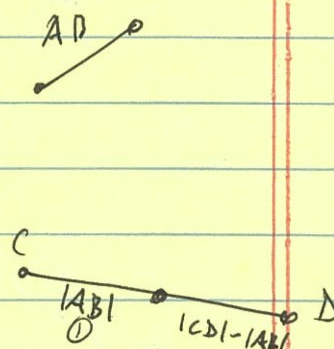
Transporting lengths:

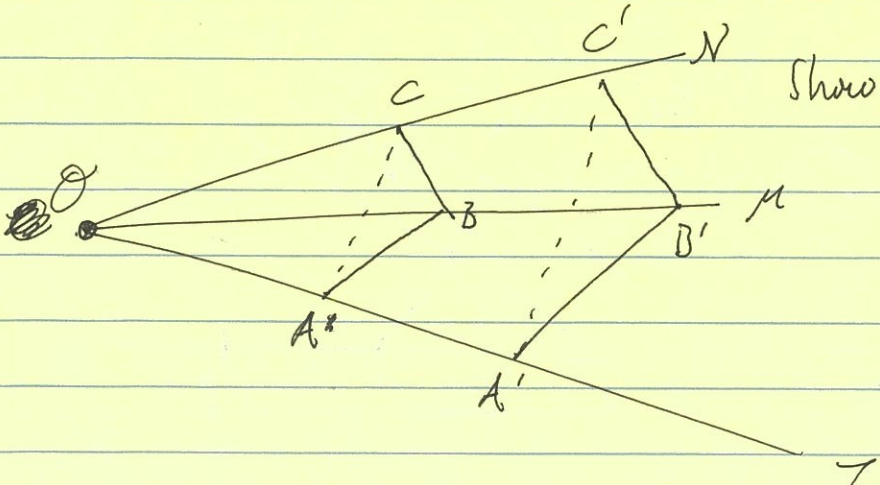


Adding:



Subtracting:





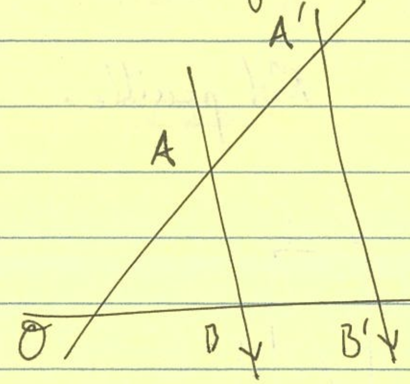
Show $AB \parallel A'B$ and $BC \parallel B'C'$

$$\Rightarrow \frac{|OA|}{|OC|} = \frac{|OA'|}{|OC'|}.$$

$$\frac{OC}{OC'} = \frac{OB}{OB'} = \frac{OA}{OA'}$$

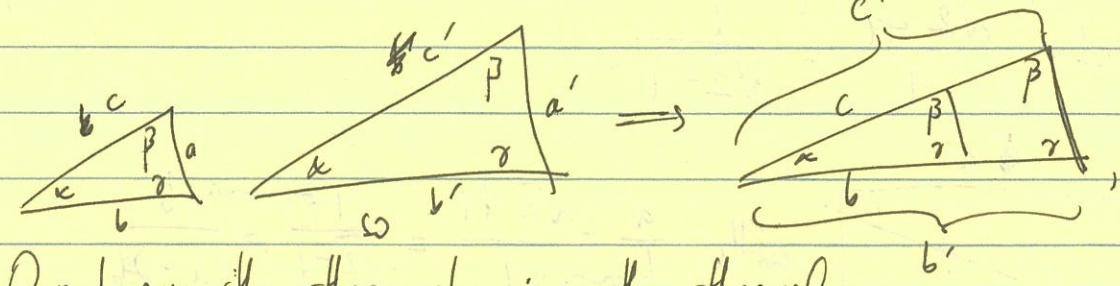
Similar triangles + irrationality. $\boxed{\text{Assign homework}}$

Thales's Thm again:



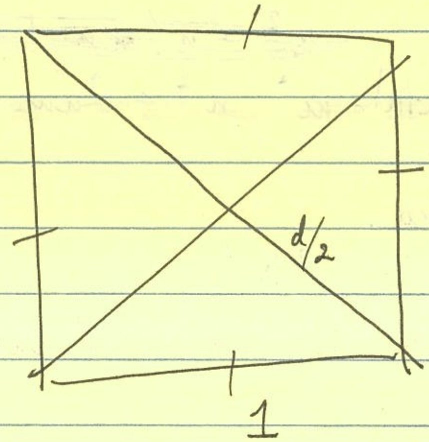
$AB \parallel A'B'$ exactly when $\dfrac{|OA|}{|OA'|} = \dfrac{|OB|}{|OB'|}$.

Similar triangles:



Overlapping the other angles gives the other sides.

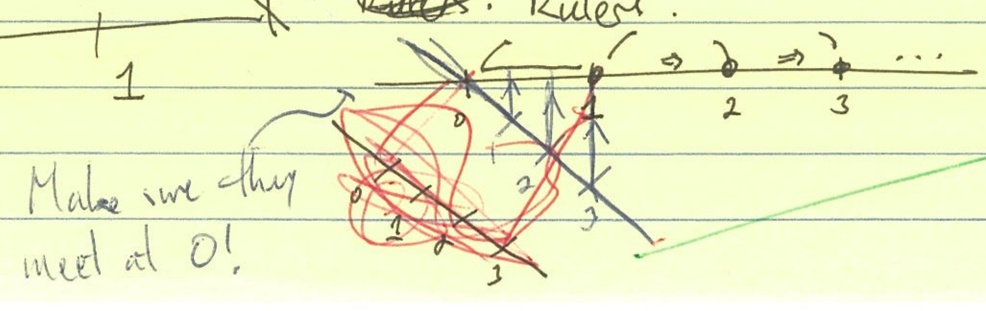
An irrational length (Bisection: same construction by using an isoceles Δ.)



One half of the square $\cong$ one triangle

$\Rightarrow \dfrac{d/2}{1} = \dfrac{1}{d}$, so $d^2 = 2$.

~~Rulers~~. Rulers:

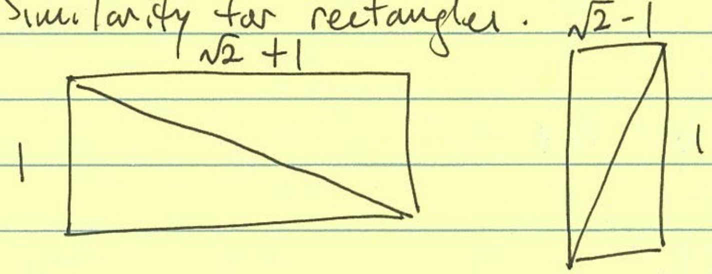


<u>Thm</u>: d satisfying $d^2 = 2$ does <u>not</u> lie on a ruler mark!

Suppose $d = \frac{m}{n}$, then $2 = \frac{d^2}{\cancel{1}} = \frac{m^2}{n^2}$, so $m = 2l$ and $m^2 = 4l^2 = 2n^2$.

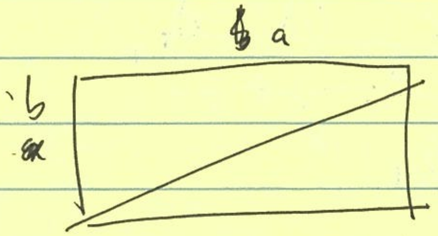
So $2l^2 = n^2$ and n is even. Not possible.

<u>Another proof</u>: Similarity for rectangles:

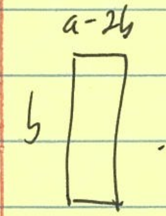


Two rectangles are proportional if the triangles so formed are proportional.

Indeed, $\frac{\sqrt{2}+1}{1} = \frac{1}{\sqrt{2}-1}$.



with $\frac{a}{b} = \frac{\sqrt{2}+1}{1} = \frac{1}{\underbrace{\sqrt{2}-1}_{=(\sqrt{2}+1)-2}} = \frac{b}{a-2b}$.



~~$m(m-2n) = n^2$~~

Now suppose $\frac{m}{n} = \frac{\sqrt{2}+1}{1} = \frac{n}{m-2n}$, so $m^2 - 2nm = n^2$.

~~$m^2 = n(m+2n)$~~

$(m-n)(m+n) = m^2 - n^2 = 2nm$

Can't both be even.

Both odd: LHS /4, RHS /2.

One even, one odd: LHS odd, RHS even. :)

Add Kat to email list

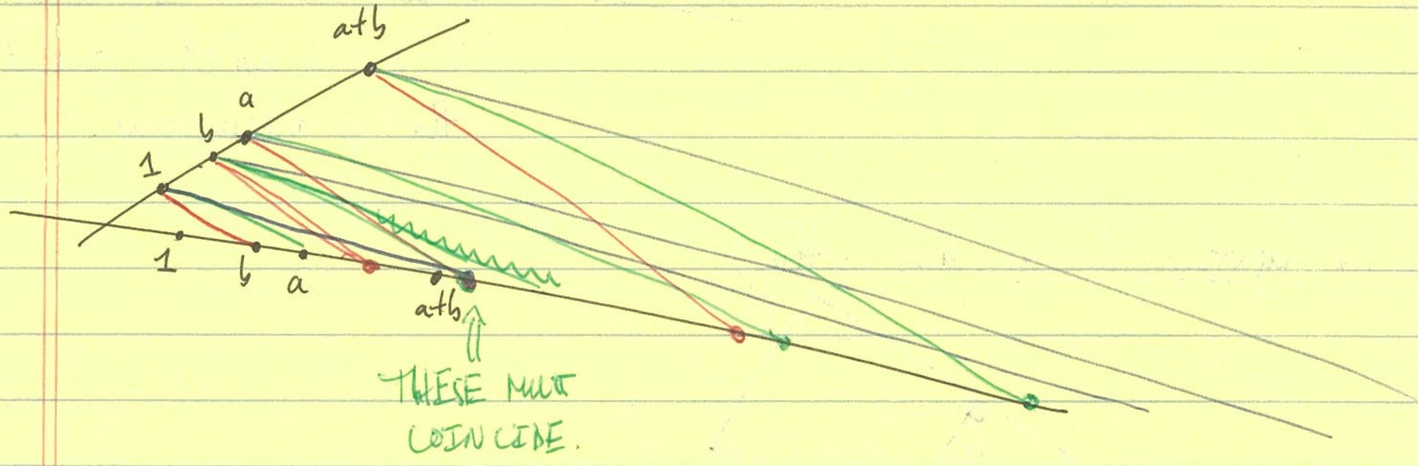
Other proof of Pythag. T.?

Mopping up from last week:

- Bisecting an angle ⇐ SSS. • Bisecting a line ⇐ SAS.
- The vertical lines of an isosceles kite are perpendicular bisectors ⇐ SAS.
- Parallel lines transport lengths ⇐ ASA. ⇐ Construct a ∥ to ℓ through P.
- Nonparallel lines distort lengths ⇐ Thales.
- ~~There are no~~ • If $\alpha+\beta=\pi$, the lines are parallel ⇐ ASA

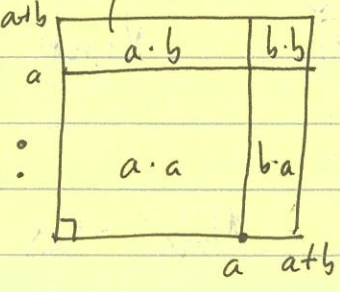
~~One more: There are no parallels~~ (If $\ell \parallel m \Rightarrow \alpha+\beta=\pi$) ⇐ unicity of geodesics

Relative strengths of numeric rep^m:



Now try to check an identity like $(a+b)^2 = a^2+b^2+2ab$...

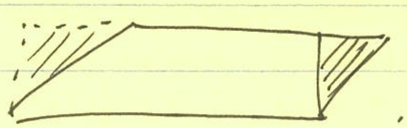
Alternatively, Areas:



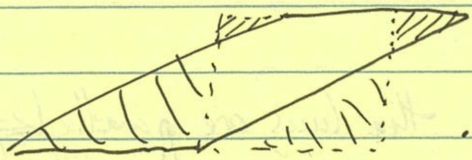
Set $A_{rect} = l\cdot w$. Then: [diagram].

Each of these are ~~squares~~ rectangles.

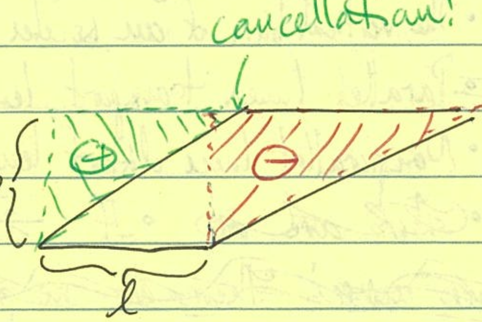
Equality + rearrangement of area: Cutting figures along straight lines preserves area.



Area of sheared parallelogram:



OR, with negative area:

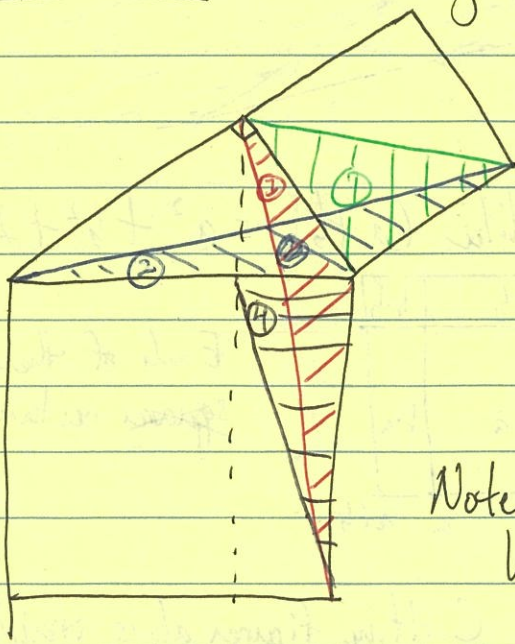


Area of parallelogram: $l \cdot w$.

Area of triangle:

$\frac{1}{2} l \cdot w$, since the two congruent Δ's make up the area of the parallelogram.

Pythagorean Theorem (more cutting):



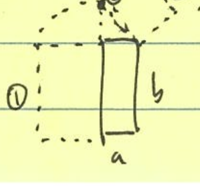
Repeat 3 times: $a^2 + b^2 = c^2$.

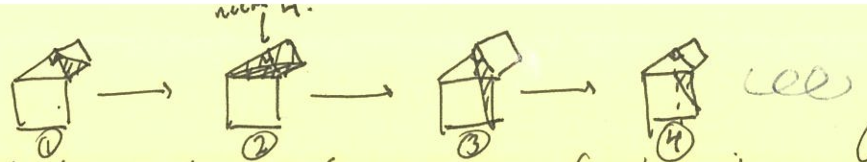
Note: This converts half a square to half a rectangle. The reverse?

This constructs a square of the same area. Its side length gives access to $\sqrt{ab}$.

We now have $+, -, \cdot, /, \sqrt{\ }$ of lengths.

③ Need to know this angle is $\llcorner$. This comes tomorrow.



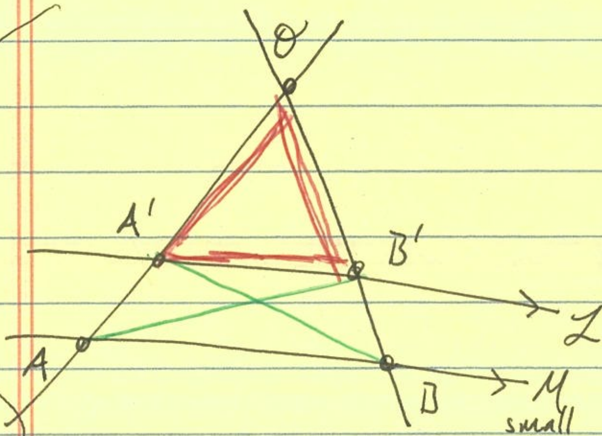


(A email)

Proof of Thales's Theorem, & using area of a triangle:

Teach this last.

It shows that the last version of the Pythag. proof is also area-dependent.

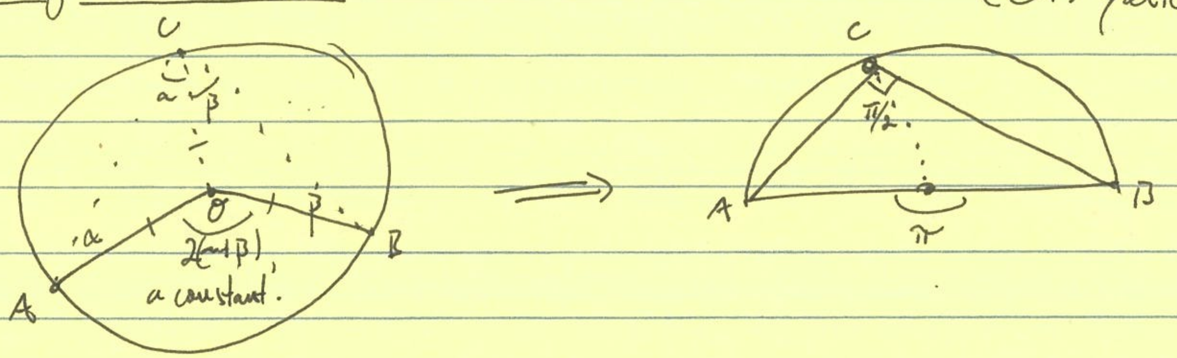


$|\Delta AB'A'| = |\Delta A'BB'|$,
because they have the same base $A'B'$ and height $d(L, M)$.
Now add $\Delta OA'B'$ to get
$|\Delta OAB'| = |\Delta OA'B|$.

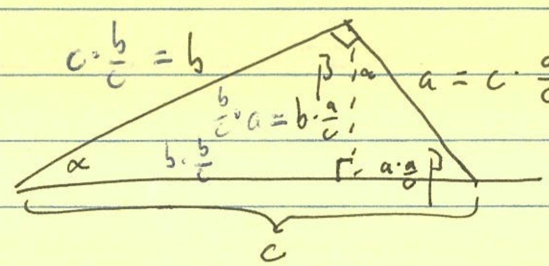
Now consider the two triangles lying on $\overline{OA}$. They have the same height (altitude from B'), so $\frac{|\Delta OA'B'|}{|\Delta AA'B'|} = \frac{\frac{1}{2}|OA'||alt|}{\frac{1}{2}|A'A||alt|} = \frac{|OA'|}{|A'A|}$

Also, $\frac{|\Delta OA'B'|}{|\Delta A'B'B|} = \frac{\frac{1}{2}|O\cancel{A}B'||alt|}{\frac{1}{2}|B'B||alt|} = \frac{|OB'|}{|BB'|}$. But these are $=$, b/c the pieces are $\cong$.

Angles in a circle: (cf. yesterday)



Pythagoras, again: (using two chords of a diameter meet at $\frac{\pi}{2}$.)



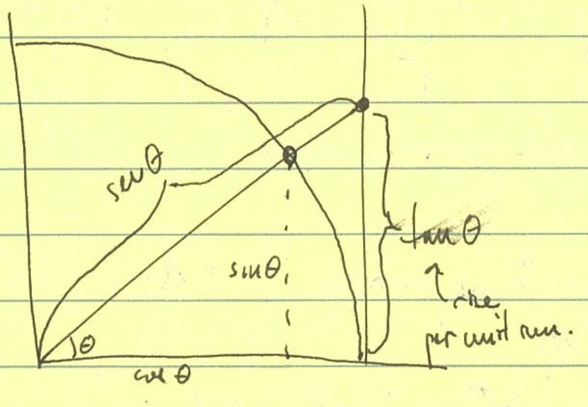
Bottom edge: $c = b\cdot\frac{b}{c} + a\cdot\frac{a}{c}$.

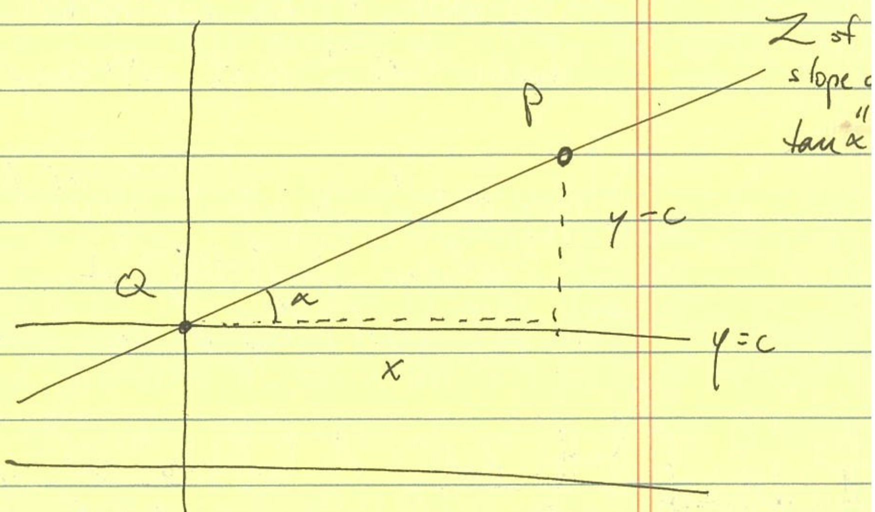
Claim: Dedekind cuts form a <u>complete</u> <u>ordered</u> field.

- Field: $+, -, \cdot, /$ all are defined. (E.g., $(A,B)+(A',B') = (A+A', B+B')$.)
- every set has a supremum = { Complete: there are no more gaps in the number line. (Every <u>Cauchy sequence</u> is convergent.) — points that cluster — cluster about another point.
- Ordered: given any two elements one of $a \le b$ or $b \le a$ is true. Moreover, these intertwine with arithmetic.

(<u>Thm</u>: $\exists$ a <u>unique</u> complete ordered field.)

Lines in coordinates:





$$\frac{y-c}{x} = a$$

$\rightsquigarrow$ $ax - y = c$.

Unless the line is vertical, then $x = c$.

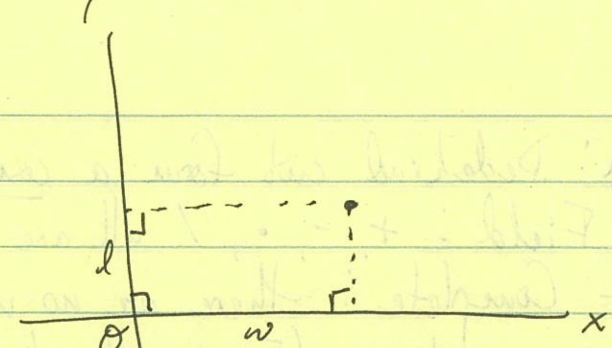
$\Longrightarrow$ $ax + by = c$, up to projective scaling.
not all zero,

Can show: ① $\exists$ unique line through any two points

② $\forall\, L\ \forall\, P$ not on L, $\exists!\ L'$ through P not meeting L.

Coordinate geometry:

Fix two lines:



Any point in the plane is specified by lengths l, w.
We have already seen it does not suffice to take l, w to be ruler values.
We have options here: take a line marked by constructible #s (but which #s are constructible?) or take a line marked so many times that it can't miss any value ($\mathbb{R}$).

Sloppy def^{n}: $x \in \mathbb{R}$ is specified by $x = \sum_{j=-\infty}^{N} 10^j \cdot c_j$, $0 \le c_j < 10$,
modulo $\sum_{j=-\infty}^{-1} 9 \cdot 10^j = 1$
This isn't great: how do we know $\sqrt{2}$ lies in here?

A is determined by B.

Better def^{n}: A Dedekind cut is a decomposition $\mathbb{Q} = A \cup B$ with
① nondegeneracy: $A \neq \emptyset$, $B \neq \emptyset$,
② A has no largest element, ③ $b \in B$ is $>$ all $a \in A$.

Ex: To represent $\sqrt{2}$, set $A = \{x \in \mathbb{Q} \mid x \le 0 \text{ OR } x^2 < 2\}$
and $B = \{x \in \mathbb{Q} \mid x > 0 \text{ AND } x^2 \ge 2\}$. $\sqrt{2}$ is the "gap" identified by $A+B$

To show this is a cut, need for all $a \in A$ $\exists$ a' with $a' > a$, $a'^2 < 2$

This only works for $a > 0$!!! $\Rightarrow$ Set $a' = \frac{2a+2}{a+2}$. Then

$$a \overset{?}{<} a' = \frac{2a+2}{a+2}$$

$$a^2 + \cancel{2a} \overset{?}{<} \cancel{2a} + 2$$

✓

$$(a')^2 \overset{?}{<} 2$$

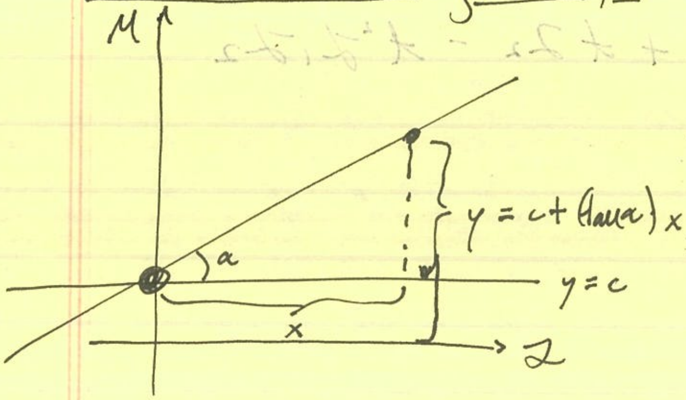
$$4(a^2 + \cancel{2a} + 1) \overset{?}{<} 2(a^2 + \cancel{4a} + 4)$$

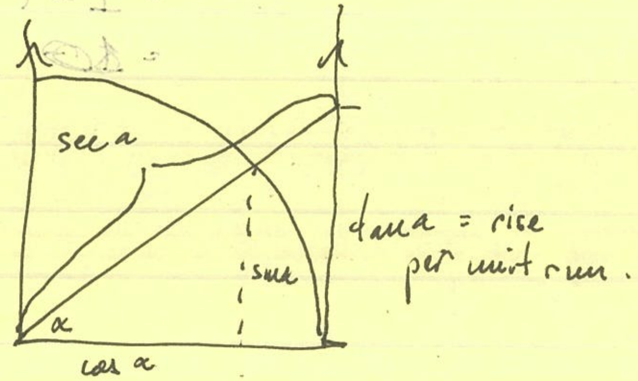
$$0 \overset{?}{<} -2a^2 + 4$$

$$0 < 4 - a^2.$$

✓

Figures ~~Lines~~ in coordinate geometry.





Works unless the line is vertical, but then $x = d$ is an eq^{n}

$\Longrightarrow$ $ax + by = c$ determines a line up to prop^{ive} eq^{ns} $[a:b:c]$

- $\exists$ a unique line through any two points.
- any line segment can be extended. • existence of parallels (same slope)

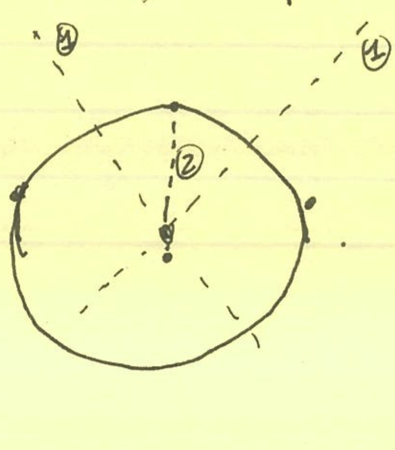
Set $d(P, Q) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$. ($\Leftarrow$ Pythagorean thm)

Get an equation for a circle: $(x - x_0)^2 + (y - y_0)^2 = r^2$.

Aside: the set of points equidistant from 2 points is a line.

Pf: Equate the two eq^{ns} for distance, cancel quadratic terms.

Show that 3 points determine a circle:



Also give a synthetic proof using angle arithmetic

Angles in coordinate geometry:

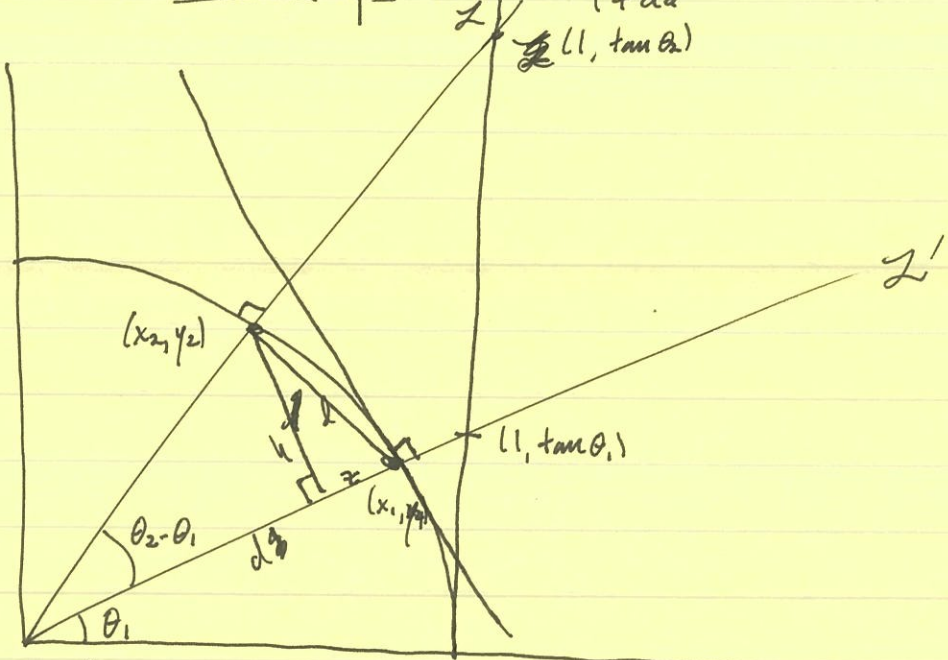
Slopes (and $\tan \alpha$) we can understand as a measure of angle.

A direct measurement of angle is more complicated, since $\tan^{-1}$ is transcendental.

On the other hand, slope is defined against reference axes.

Claim: If $\mathcal{L}$ has slope a and $\mathcal{L}'$ has a',
then the relative slope is $\pm \frac{a-a'}{1+aa'}$.

Pf:



$l^2 = 2(1 - x_1 x_2 - y_1 y_2)$. Also, $d^2 + h^2 = 1$ and $h^2 + z^2 = l^2$

$h^2 + (1-d)^2 = l^2$

Can solve for d from here, in terms of the x's and y's.

Then, just use $d^2 + h^2 = 1$ to get h. ← this is hard.

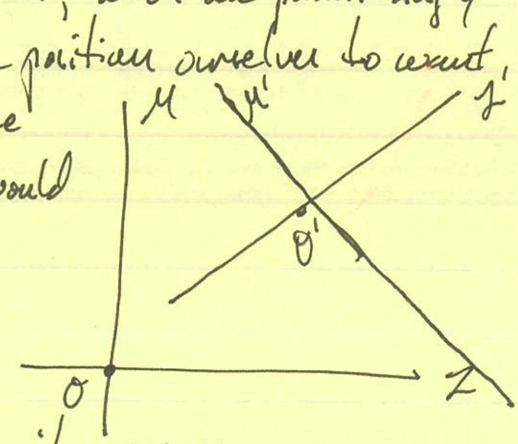
Altogether, $\tan(\theta_1 - \theta_2) = \dfrac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \cdot \tan\theta_2}$.

Consequence: Perpendicular lines have slopes which are opposite reciprocals.

HW #3 is up
Term paper topics: 2/22.
Email me!

Isometries:

Euclid used "rigid motion" in some of his proofs, which we painstakingly proved was reasonable. We are now in the position ourselves to want to use rigid motions: our coordinate plane depends upon a *choice* of axes, and it would be nice to claim something like "any two ruled axes can be moved into one another".

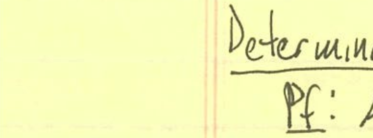


Defn: $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an *isometry* when it preserves distances: $d(P,Q) = d(f(P), f(Q))$!

Examples: translations, rotations, and reflections (and glide reflections)
preserves too many lines; fixes *one* point; fixes *many* points.

Determination theorem: An isometry is determined by its behaviour on 3 noncollinear pts.
Pf: Any point is determined by its distances from A, B, C.
(If there were 2 points, ABC would be equidistant $\Rightarrow$ collinear.)
Use this fact on $f(A), f(B), f(C)$ from $f(x)$.

3 reflections theorem: Any isometry of $\mathbb{R}^2$ is a composition of 1, 2, or 3 reflections.
Pf: Reflect A to $f(A)$ by the perpendicular bisector: r_A.
If B is carried to $f(B)$, do nothing. Otherwise, send $r_A(B)$ to $f(B)$ by reflection across the equidistant line of $r_A(B)$ and $f(B)$ — and this preserves $f(A)$, since it lies on the line, as $|\overline{f(A)f(B)}| = |\overline{f(A)r_A(B)}| = |\overline{r_A(A)r_A(B)}| = |\overline{AB}|$.
Do something similar for C.

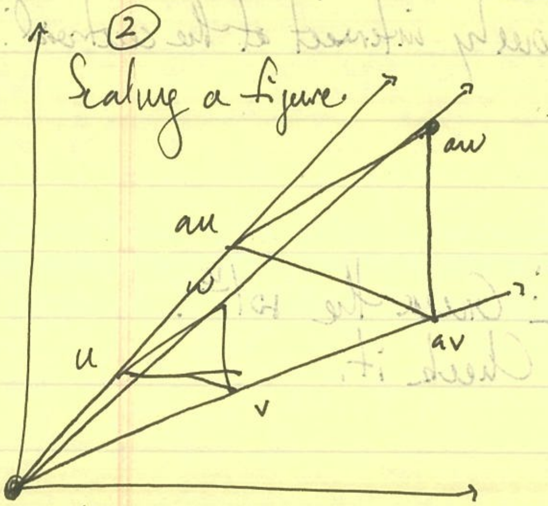


Homework: 1: reflections, 2: translations+rotations, 3: glide reflections.
Linear alg. Curves of degree d. Isometries are groups.

Vector Geometry

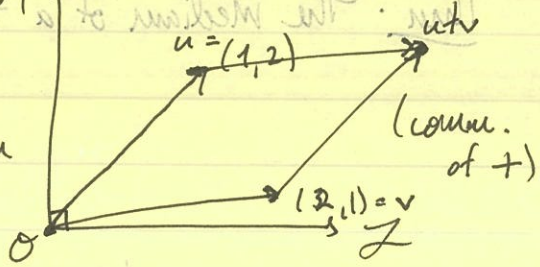
A *vector space* is a context where *vectors* can be *added*, *subtracted*, and *scaled*. For example: $\mathbb{R}^2$.

② Scaling a figure

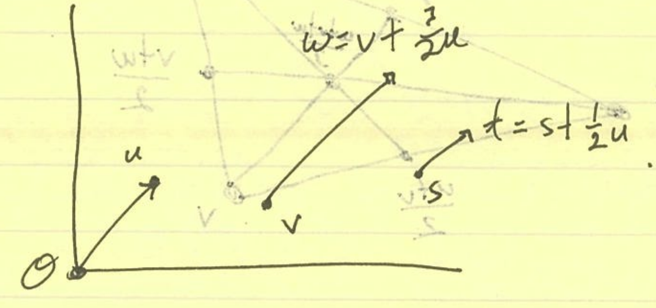


(Desargues's theorem!)

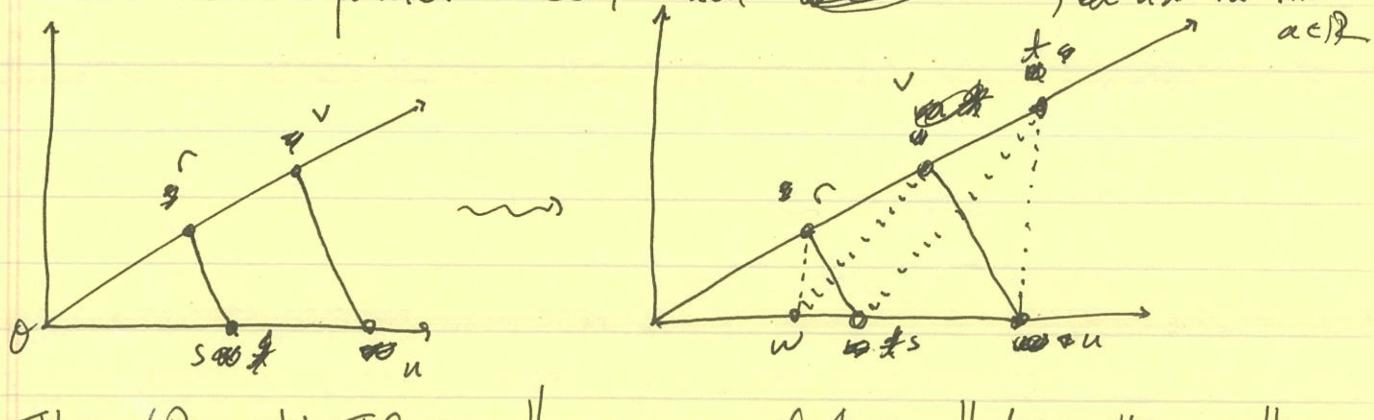
① Parallelogram Law of Addition



③ Parallel vectors



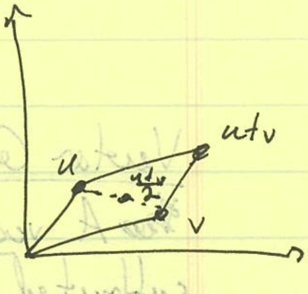
Thm (Thales): If r, v are on a line through 0 and so are s, u, and $r - s$ is parallel to $v - u$, then $v = ar$, $u = as$ for fixed $a \in \mathbb{R}$



Thm (Pappus): If $r - s \parallel v - u$ and $r - w \parallel t - u$, then $v - w \parallel t - s$.

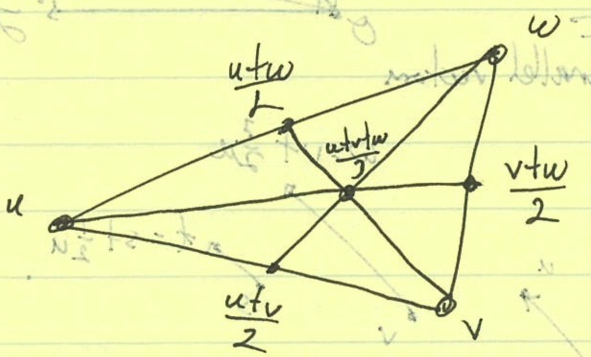
CONTINUE ON BACK.

Midpoints (before distance): $u + \frac{1}{2}(v-u) = \frac{u+v}{2}$.



Note that this bisects two lines.

Thm: The medians of a triangle simultaneously intersect at the centroid:



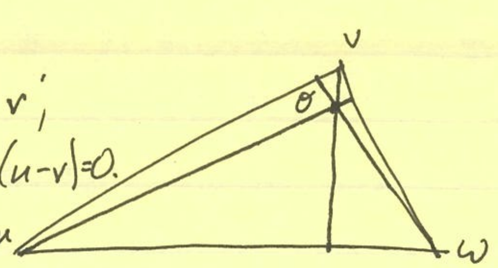
Pf: Guess the sol^{n}. Check it.

The inner product

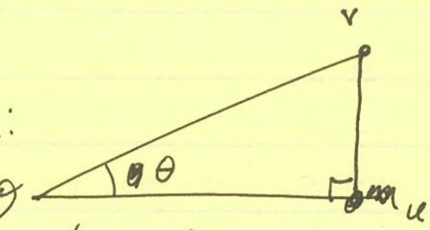
Def: $\begin{pmatrix}u_1\\u_2\end{pmatrix}\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix} = u_1v_1+u_2v_2$.

Note: ① The length of u is $\sqrt{u_1^2+u_2^2} = \sqrt{u\cdot u}$.
More generally, $|u-v|^2 = u\cdot u + v\cdot v - 2u\cdot v$.
② Vectors are perpendicular exactly when $u\cdot v = 0$.
(This follows from $\tan(\theta_2-\theta_1) = (\tan\theta_1+\tan\theta_2)/(1+\tan\theta_1\tan\theta_2)$
$\Longrightarrow \tan\theta_1 = (-\tan\theta_2)^{-1}$.)

Lem: Altitudes in a triangle coincide.



Pf: Intersect the altitudes for u and v; place that at 0. Need to show $w\cdot(u-v)=0$.
Have $u\cdot(v-w)=0$ and $v\cdot(w-u)=0$.
Add.

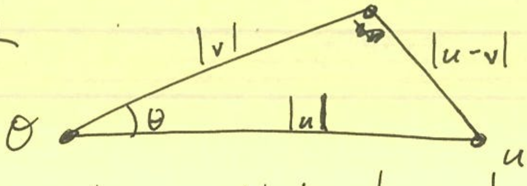


Def: Suppose u and v form a right triangle:
Set $\cos\theta = |u|/|v|$.

~~More generally~~ Then, $u\cdot v = |u||v|\cos\theta$. (Rescale u down if necessary.)

Pf: $v-u$ is $\perp$ to u, so $(v-u)\cdot u = 0 = u\cdot v - u\cdot u$.
So, $u\cdot v = u\cdot u = |u|^2 = |u|^2\cdot\frac{|v|}{|v|} = |u||v|\cos\theta$.

Cor:

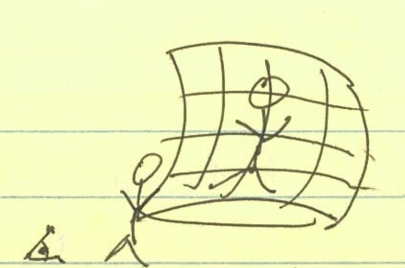


$\rightsquigarrow |u-v|^2 = |u|^2+|v|^2-2|u||v|\cos\theta$.

The triangle inequality: $|u\cdot v| \le |u||v|$ (since there's no cosine).

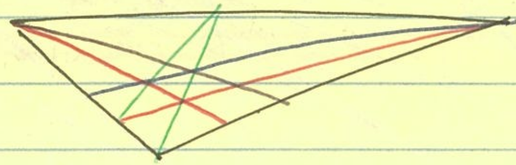
$$|u+v|^2 = (u+v)\cdot(u+v) = u\cdot u + v\cdot v + 2u\cdot v$$
$$\le |u|^2 + 2|u||v| + |v|^2 = (|u|+|v|)^2.$$ Take roots.

Projective geometry

④ 

Projective geometry.

②



① ② ③ ④ ...

Key ingredients: A horizon line. A straightedge.

Key ~~axiom~~ observations: Straight lines remain straight (regardless of perspective point).
Intersections remain intersections. Parallel lines remain $\parallel$ or meet at ∞.
(If you rotate your perspective, all parallel lines meet at ∞.)

Axioms: Any two points give a unique line. Any two lines meet at a unique point. There are ≥ 4 points, no 3 of which lie on a line.

★ (This last one bootstraps the tiled plane construction.) ★

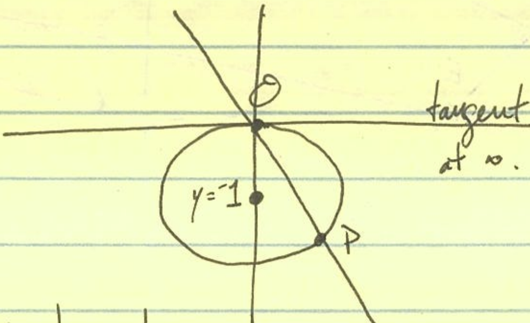
Check these axioms!

A model for projective geometry:

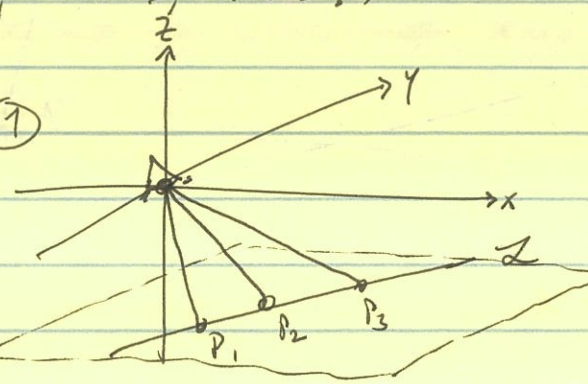
- Points are lines through O.
- Lines are planes through O.
- (Consider $[0:0:1]$, $[0:1:0]$, $[1:0:0]$, $[1:1:1]$.)

This is notation for scalar multiples.

②

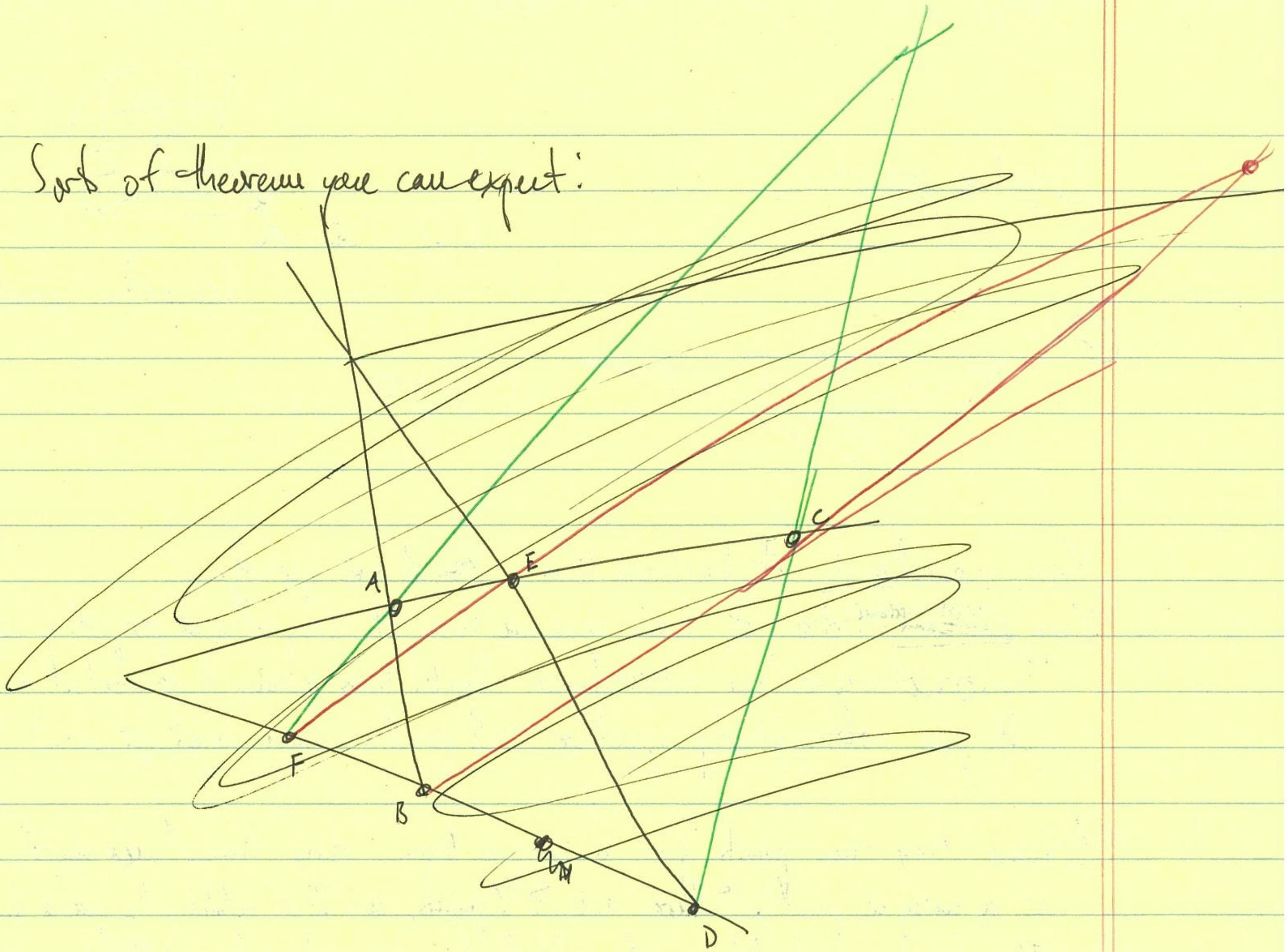


①

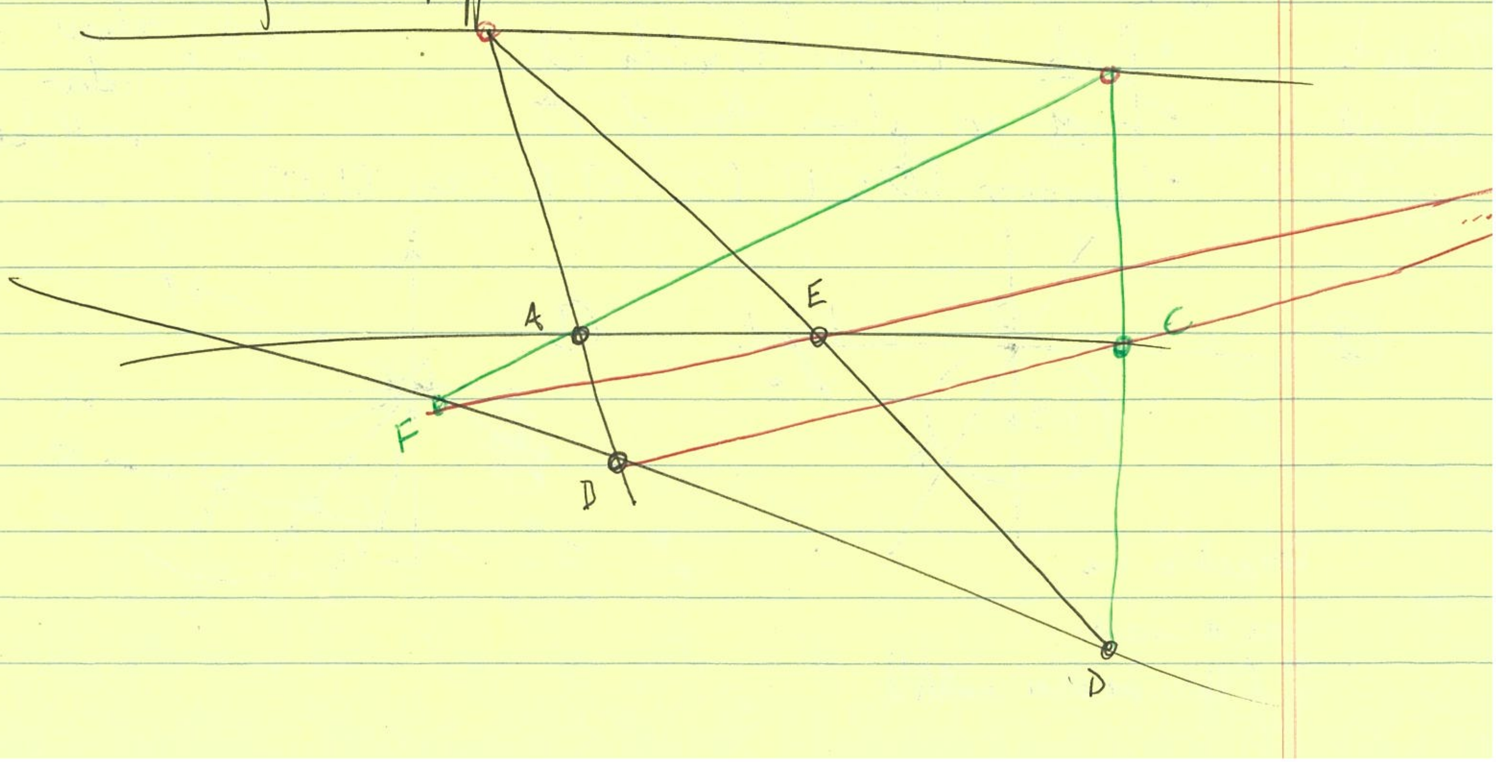


Projective line, as a circle
(There are other models!)

Sort of theorem you can expect:



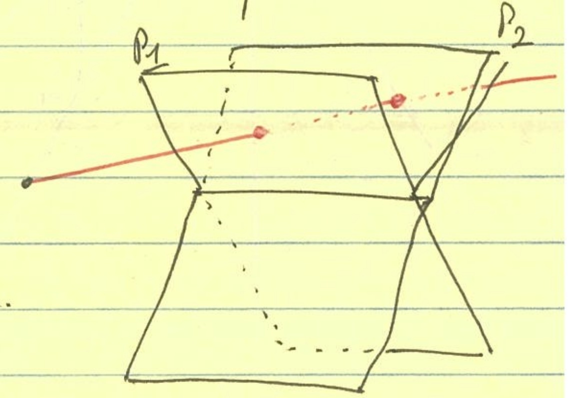
Projective Pappus's theorem.



Picturing $\mathbb{RP}^2$: Set $c=1$, consider $[a:b:1]$. This is an $\mathbb{RP}^1$.
What are the points near this ~~seat~~ seam? They form a mobius band:
if you're on the seam, you come back to where you started,
but if you're off by ε, you come back to $-\varepsilon$.

Picturing the "bulk" of $\mathbb{RP}^2$:
Pick a plane not containing the origin. All points except one line (the parallel plane) project onto this plane.

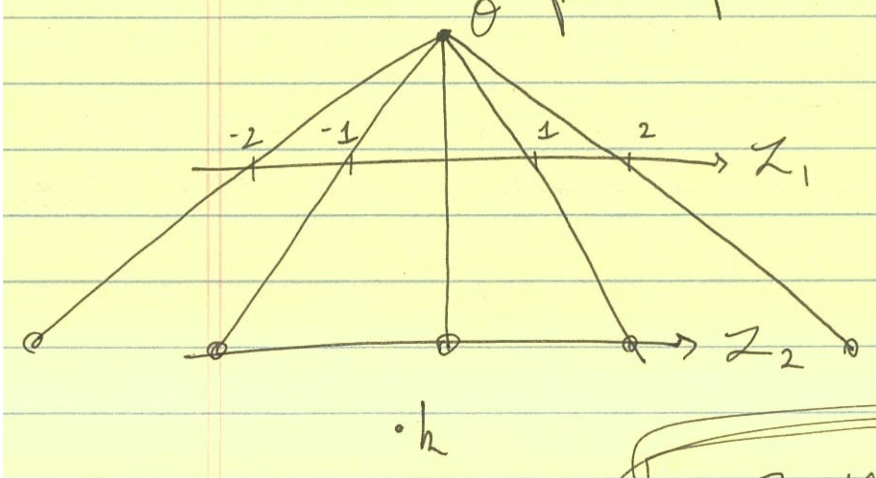
Consider two such planes:

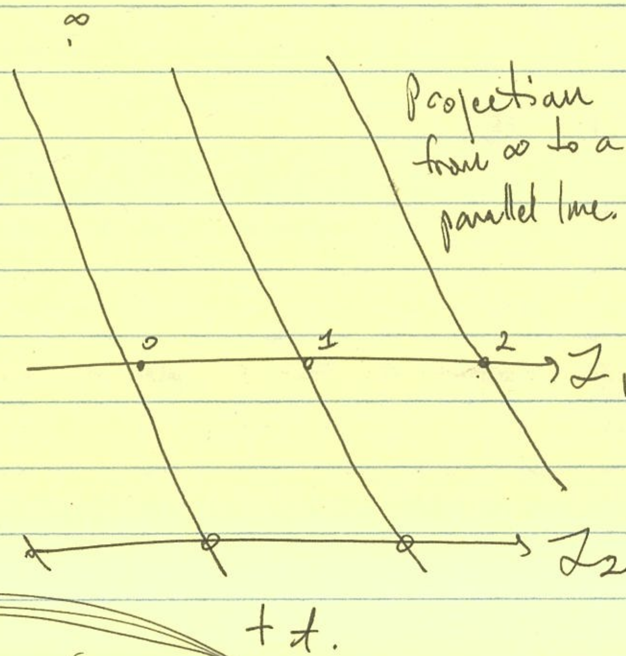


The transformation of points from one projection to another causes distortions.
Q: What kinds of distortions?

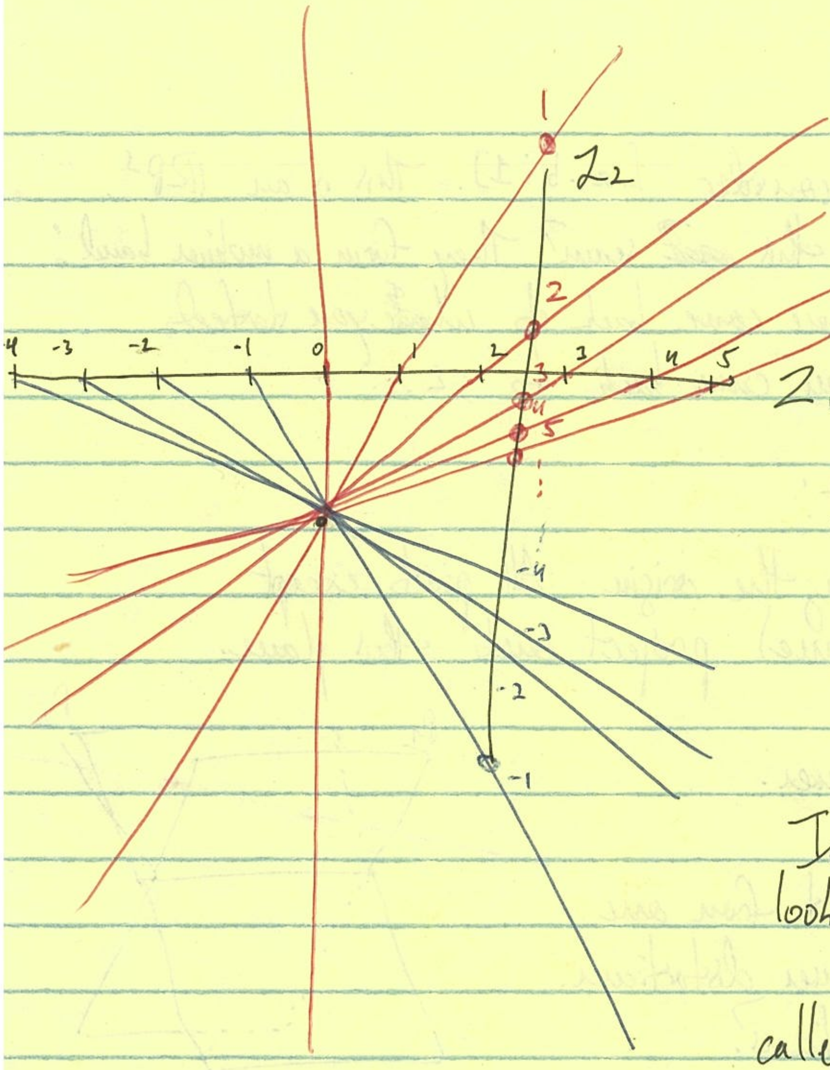
Simpler case: $\mathbb{P}^1$:

Projection from a finite point to a parallel line.





Build formulae??



Projection to a non-∥ line from a finite point.

$\frac{1}{x}$.

In general, any transformation looks like $\frac{ax+b}{cx+d}$; called Linear Fractional Transformations or Möbius Transformations.

5.6

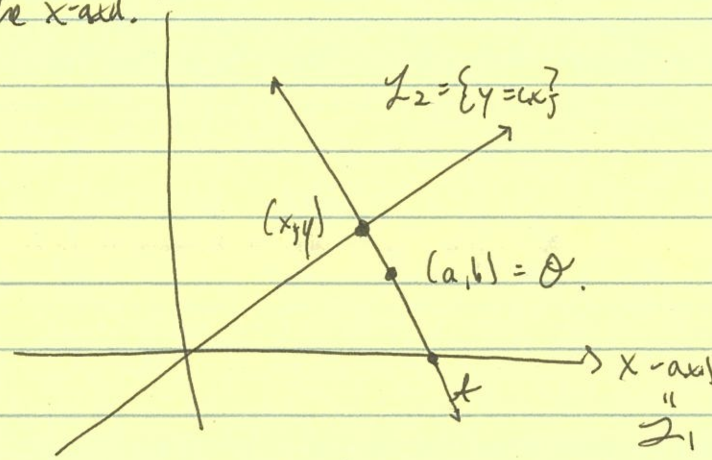
LFTs and $\mathbb{RP}^1$:

Start with $f(x) = \dfrac{ax+b}{cx+d} = \dfrac{\frac{a}{c}(cx+d) - \frac{a}{c}d + b}{cx+d} = \dfrac{a}{c} + \dfrac{-ad+bc}{c(cx+d)}$.

So, any LFT decomposes as the kinds we saw last time.

Q: Does any seq$^{\underline{ce}}$ of projections yield an LFT?

Assume one of the lines is ~~on~~ the x-axis.



$\dfrac{a-x}{b-y} = \dfrac{a-t}{b}$, but $y=cx$ gives

$\dfrac{a-x}{b-cx} = \dfrac{a-t}{b} \rightsquigarrow x = \dfrac{bt}{ct-ac+b}$

Now: • Composing LFTs $\Rightarrow$ LFT.
• Inverting LFTs $\Rightarrow$ LFT. } these form an iso$^{\underline{mety}}$ gp!

$\Rightarrow$ Any projection gives an LFT.

Observations: ~~makes sense globally as a map $\mathbb{RP}^1 \to \mathbb{RP}^1$.~~

• LFTs witness some "flexibility" of $\mathbb{RP}^1$.

a) $\mathbb{R} \longrightarrow \mathbb{R}$ an isometry is determined by translation + reflection.

b) $\mathbb{R} \longrightarrow \mathbb{R}$ a projection is determined by translation + dilation.

c) $\mathbb{RP}^1 \to \mathbb{RP}^1$ is determined by a, b, c, d up to projective eq$^{\underline{ce}}$.

(So, e.g., $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto 3$ is allowed, but then all other values are automatically determined.)

"The three point rule."

Show this by showing $(r,s,t) \mapsto (0,1,\infty)$ is possible, and inverting.

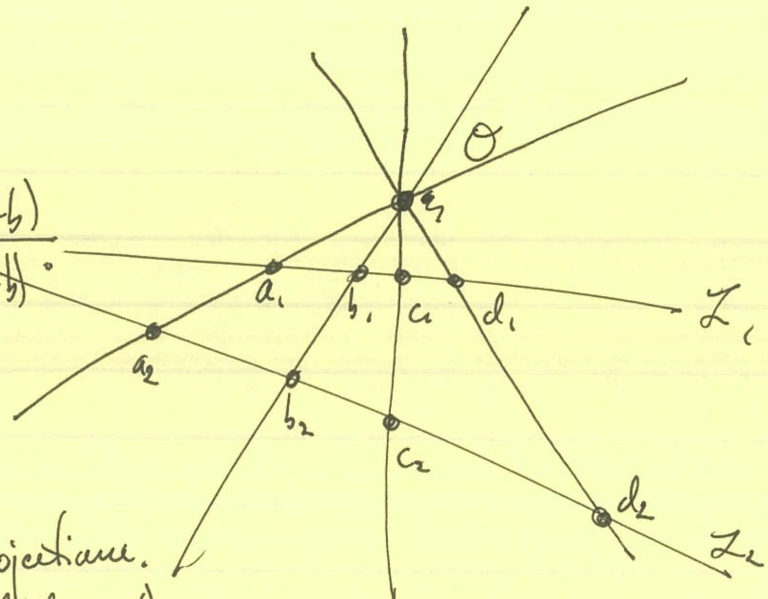
Cliff Taubes

Proj$^{\underline{ns}}$ Seq$^{\underline{ces}}$ of std Proj$^{\underline{ns}}$

LFTs $\longrightarrow$ Seq$^{\underline{ces}}$ of std LFTs

The cross-ratio (in $\mathbb{RP}^1$):

$$\frac{(c-a)/(d-a)}{(c-b)/(d-b)} = \frac{(c-a)(d-b)}{(d-a)(c-b)}.$$



Claim #1: Cross-ratio is preserved under re-projection.

Pf: Check the 3 standard projections.

Translation: $\dfrac{(c+t-a-t)/(d+t-a-t)}{(c+t-b-t)/(d+t-b-t)}$.

Scaling: $\dfrac{(kc-ka)/(kd-ka)}{(kc-kb)/(kd-kb)}$.

$\perp$: $\dfrac{\left(\frac{1}{c}-\frac{1}{a}\right)/\left(\frac{1}{d}-\frac{1}{a}\right)}{\left(\frac{1}{c}-\frac{1}{b}\right)/\left(\frac{1}{d}-\frac{1}{b}\right)} \left(\cdot \dfrac{cd\,\frac{a}{a}}{cd\,\frac{b}{b}} \right).$ $\square$

Claim # 2: Given 3 pts, + a cross-ratio, there is a unique 4th point realizing the ratio.

Pf: Just solve for d.

~~Cumulative claim~~: OR: the map $\frac{(x-p)(r-q)}{(x-q)(r-p)}$ sends (p,q,r) to $(0,1,\infty)$, and it sends x to its cross-ratio. Invert this LFT.

~~Claim #3~~:

Cumulative claim: LFTs are the unique class of transformations preserving the cross-ratio.

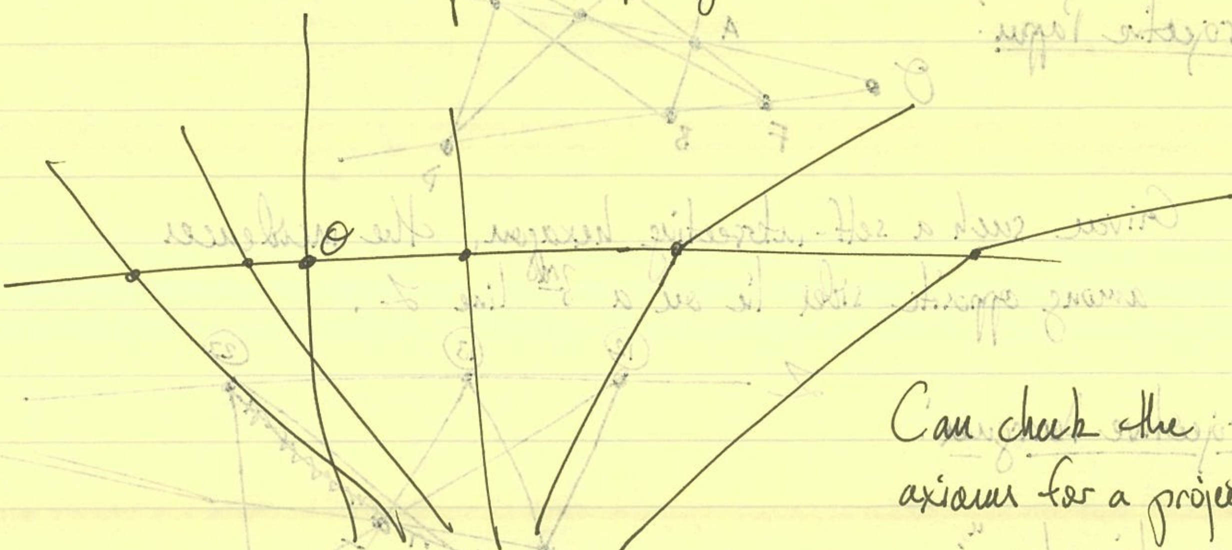
Augmented claim: All numerical invariants of 4 pts of LFTs are cross-ratios.

Pf: Any quadruple of points w/ ratio R can be carried to any other w/ ratio R by an LFT. So, I becomes a f^n of the x-ratio!

double the angle?
Yeah, OK.

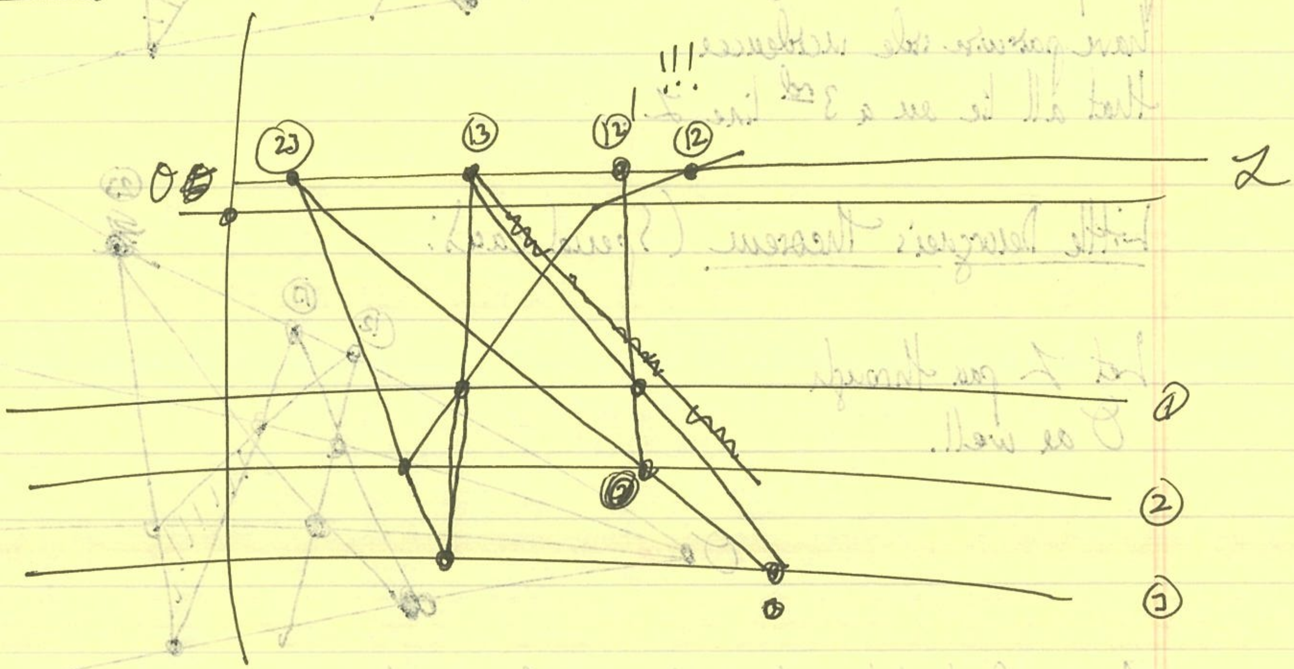
Moulton's Plane:

Lines of positive slope get bent across the x-axis.



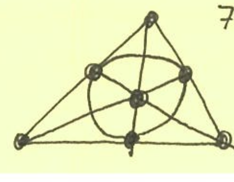
Can check the axioms for a projective plane.

However:



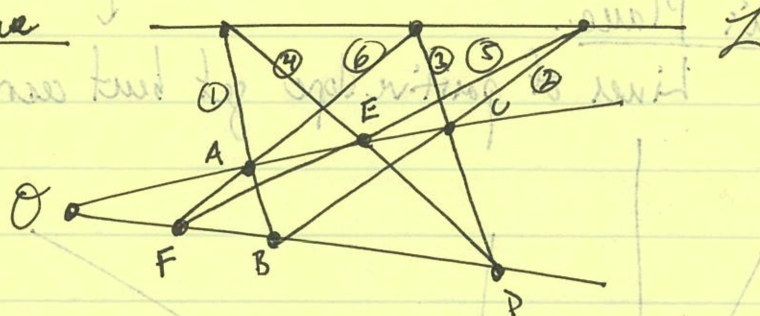
Proof of Pappus in $\mathbb{RP}^2$: Reproject so that 12 and 23 lie on L_∞. Then we want 13 to lie on ∞. This is the usual Euclidean Pappus, which we proved in vector geometry.

7 pts



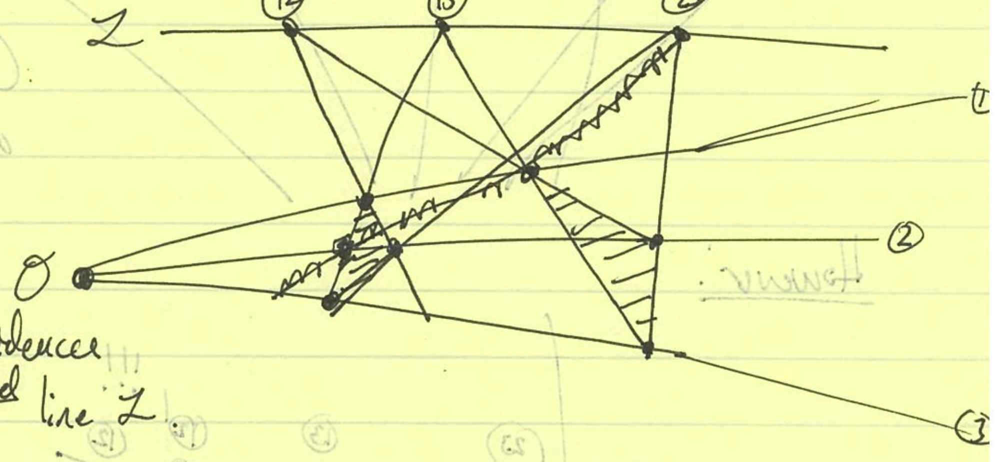
The Moulton Plane

Projective Pappus:



Given such a self-intersecting hexagon, the incidences among opposite sides lie on a 3^{rd} line $\mathcal{L}$.

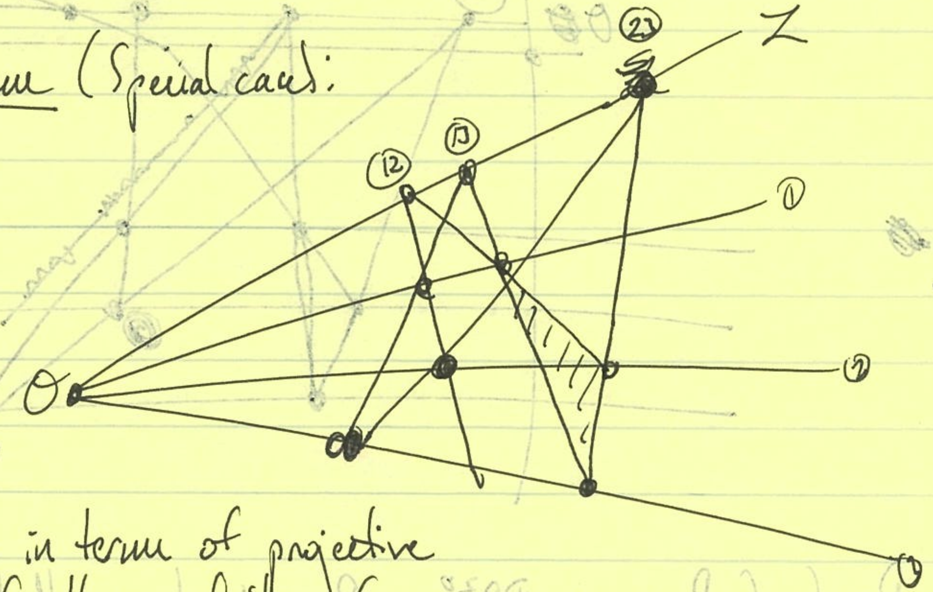
Projective Desargues:



Two triangles "in perspective from O" have pairwise side incidences that all lie on a 3^{rd} line $\mathcal{L}$.

Little Desargues's Theorem (Special case):

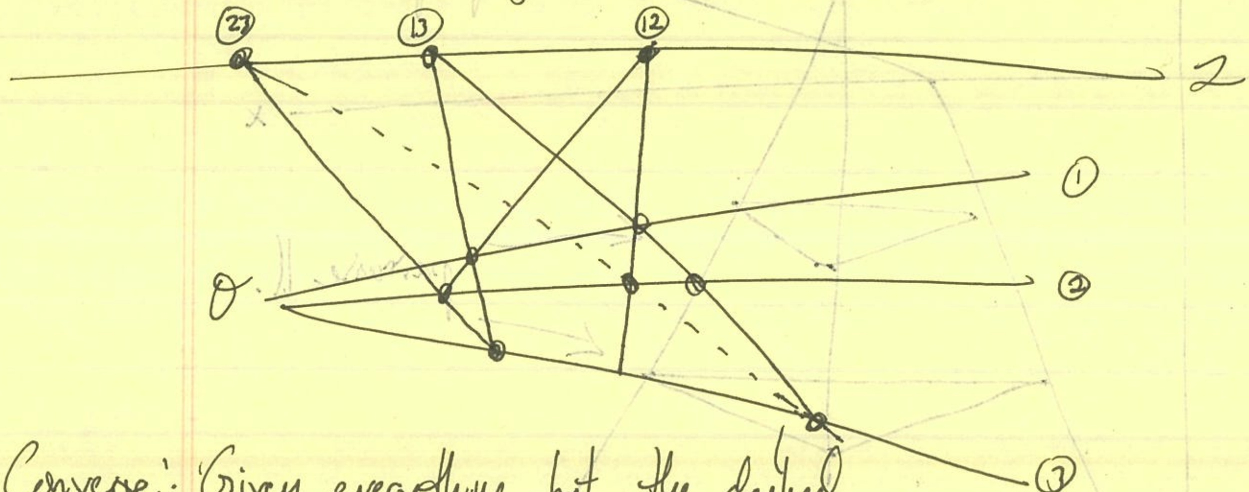
Let $\mathcal{L}$ pass through O as well.



These admit statements in terms of projective geometry. None of them admit proofs.

~~Proof of Pappus in $\mathbb{RP}^2$~~

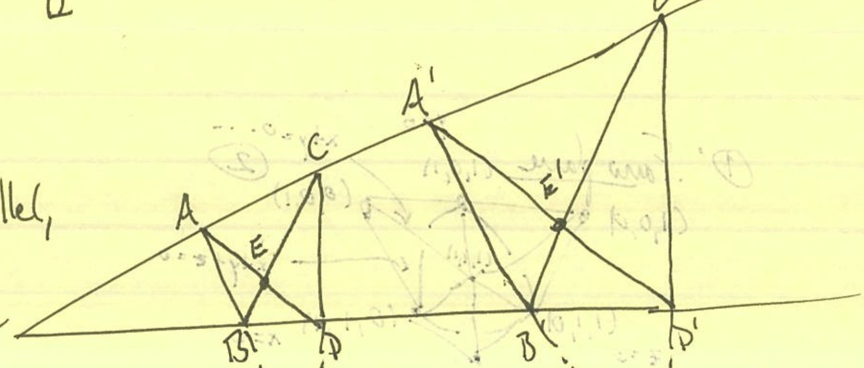
More on Desargues's



<u>Converse</u>: Given <u>everything</u> but the dashed line, (23)2' meets 3'.

<u>Pf</u>: Suppose (3) meets $\overline{(23)2'}$ at some other point 3''.
Then $\triangle 123$ and $\triangle 1'2'3''$ are in perspective from O.
Desargues + assumptions about $\overline{12} \cap \overline{1'2'}$ and $\overline{13} \cap \overline{13''}$ meeting on L
forces $\overline{23} \cap \overline{2'3''}$ on L too. So (23)2' meets 3'' on L and 3' on L.
This means $3' = 3''$. ◻

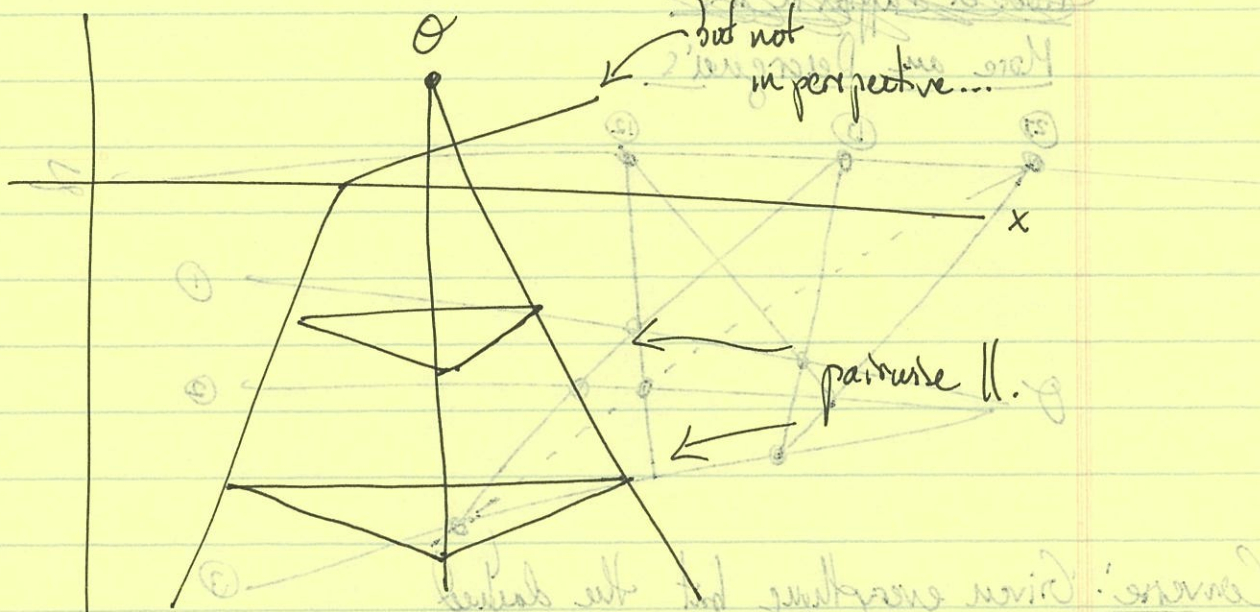
<u>Scissors' Theorem</u>:
<u>If</u> 3 pairwise sides are parallel,
then so is the <u>4th</u>.
(Say CD, C'D') O



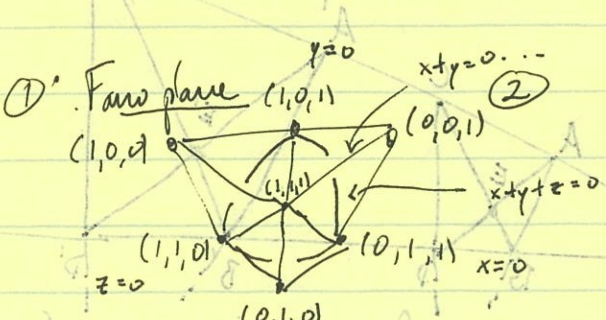
<u>Pf</u>: ABE and A'B'E' form a Desargues configuration in persp. from O.
It follows that CDE and C'D'E' are in perspective too.
Desargues says CD $\parallel$ C'D'.

<u>Pf</u> (Say BC and B'C'): Extend AB*, CD and A'D', C'D' so they meet.

Moulton, again:



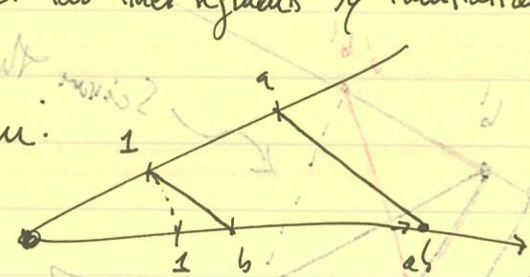
$\Longrightarrow$ Converse Desargues' fails.



Arithmetic

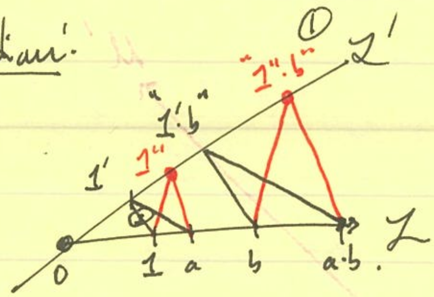
Euclidean: Connect two lines segments by translation.

Multiply using Thales's theorem:



In projective geometry, we want to draw analogous pictures w/o compass.

Multiplication:



① Pick another line $\mathcal{L}'$.

② Mark it somewhere with 1'.

③ Connect to a and 1.

④ Draw the ∥ figure at b.

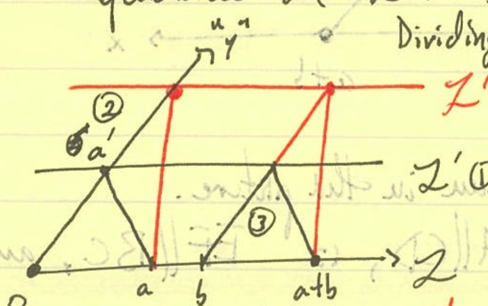
Question: Dependence on choice of 1'? ← Scissors!

Dependence on order: a·b vs b·a? Associativity?

Dependence on $\mathcal{L}'$? Dependence on the primes version vs originals?

Dividing?

Addition:



① Pick a parallel line $\mathcal{L}'$, so that length is not distorted.

② Pick another point on it, 0'.

③ Build the translated △.

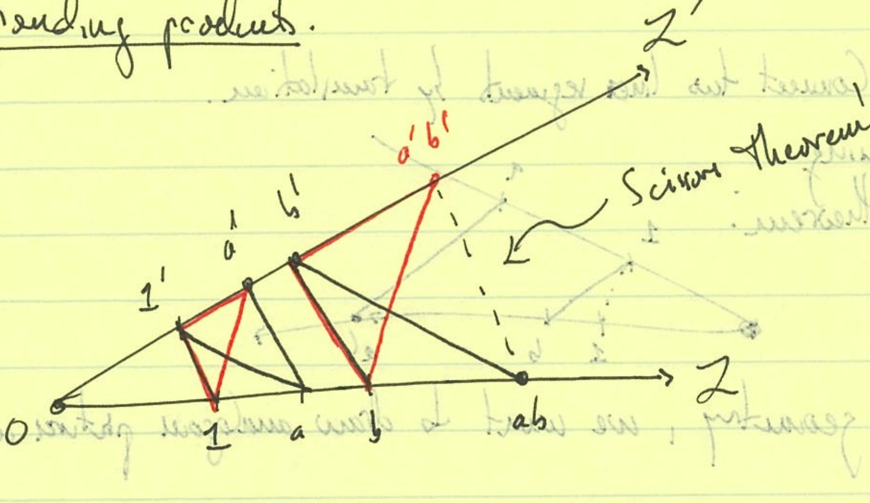
Desargues's!

Question - Dependence on $\mathcal{L}'$? Dependence on a'?

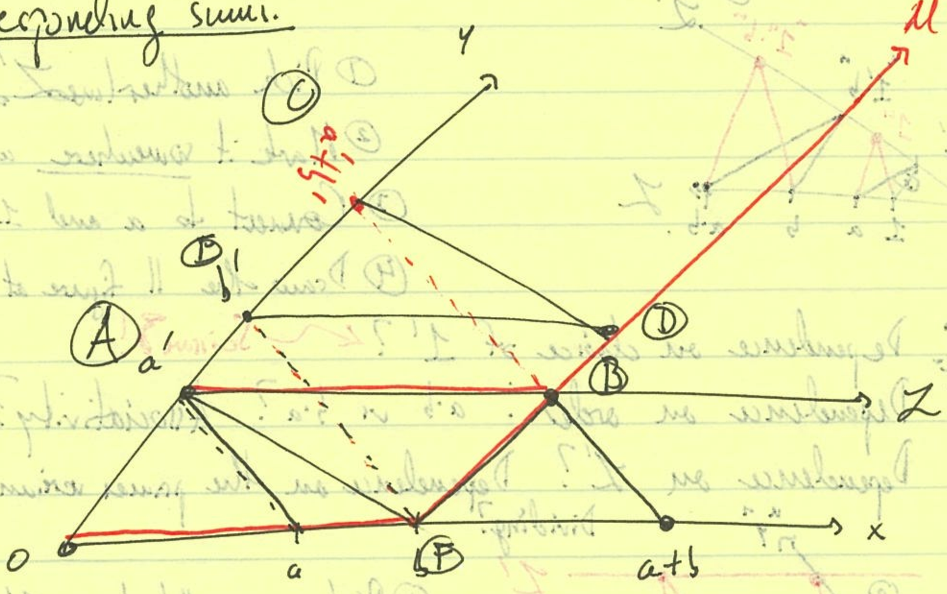
Dependence on order: a+b vs b+a? Associativity? Distributivity?

Dependence on primes vs. originals? Subtraction?

Corresponding products:



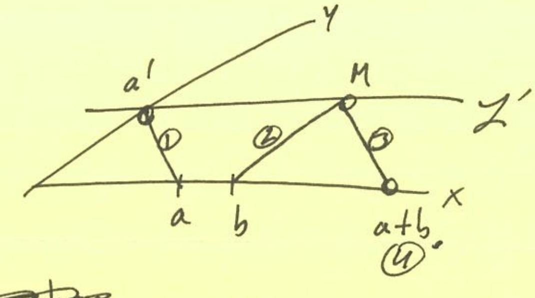
Corresponding sums:



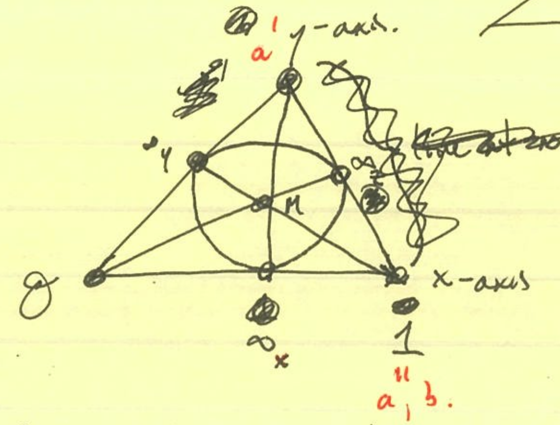
There is a Pappus configuration in the picture.

AB||DE and FA||CD, so EF||BC, and a'+b', B, a+b are collinear.

Examples in ~~Fano~~ arithmetic



An example:



circle = line at ∞.
vertical line = $\mathcal{L}'$.

① a' is a noninfinite, ~~point~~ nonzero point on y.
② The far edge of the Δ is the line aa'.
③ Construct the $\parallel$ Δ off of b.

$\Delta\, 0 a' a$ is parallel to ~~...~~

$\Delta\, b\, M\, 0$ ← the $\parallel$ to $a'a$ is $0M\infty_z$, meets $0a$ at 0.

start at b

the $\parallel$ to $0a'$ is $1M\infty_y$, meets the ~~...~~ vert. line at M

So, $1+1=0$ in Fano arithmetic, as predicted by $\mathbb{F}_2$.
(This used every point + every line, so maybe Fano is minimal...?)

<u>Other example</u>: $\mathbb{H}P^2$, the projective space for a division algebra.

Write $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. "Pauli"

Then $\mathbb{H} = \mathrm{Span}_{\mathbb{R}}\left(\overset{1}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \overset{i}{-i\sigma_1}, \overset{j}{-i\sigma_2}, \overset{k}{-i\sigma_3} \right)$. $a\cdot 1 + b\cdot \underline{i} + c\cdot \underline{j} + d\cdot \underline{k}$

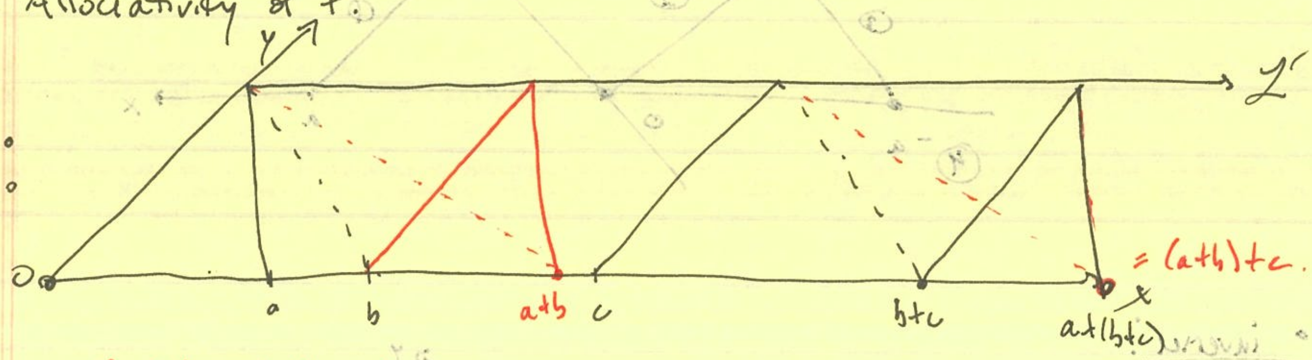
The subgp of unit norm forms "$SU(2)$", the norm-preserving $\mathbb{C}$-linear automorphisms of $\mathbb{C}^2$.

Can form $\mathbb{H}P^2$ in the same way: $(\mathbb{H}^3 \setminus 0)/\mathbb{H}^\times$, where $\mathbb{H}^\times$ acts on <u>one side</u>. This satisfies Desargues's, but not Pappus.

Field axioms in Pappian arithmetic:

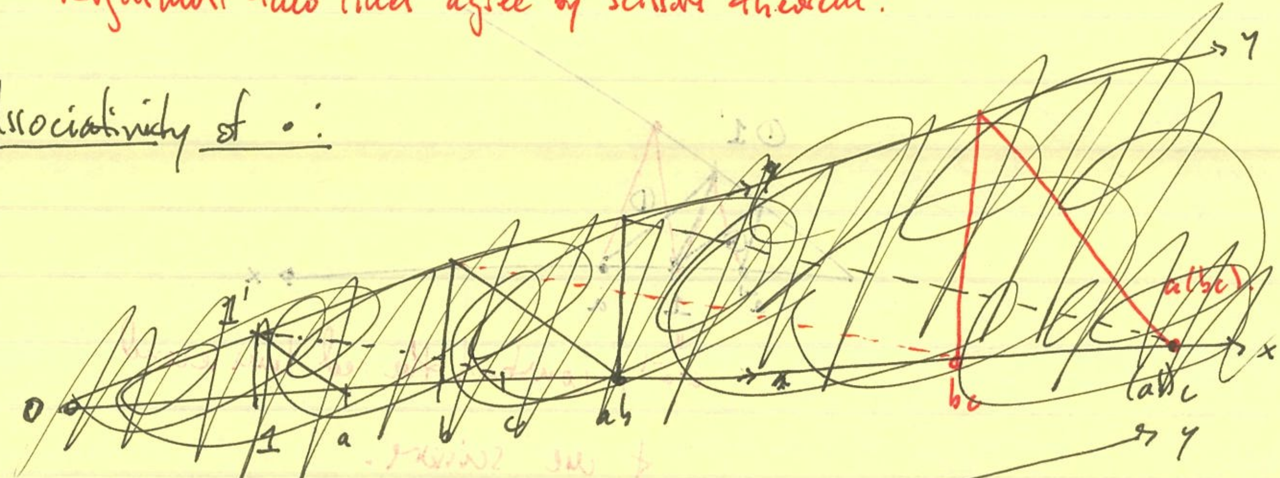
Associativity of +:

Little Desargues.

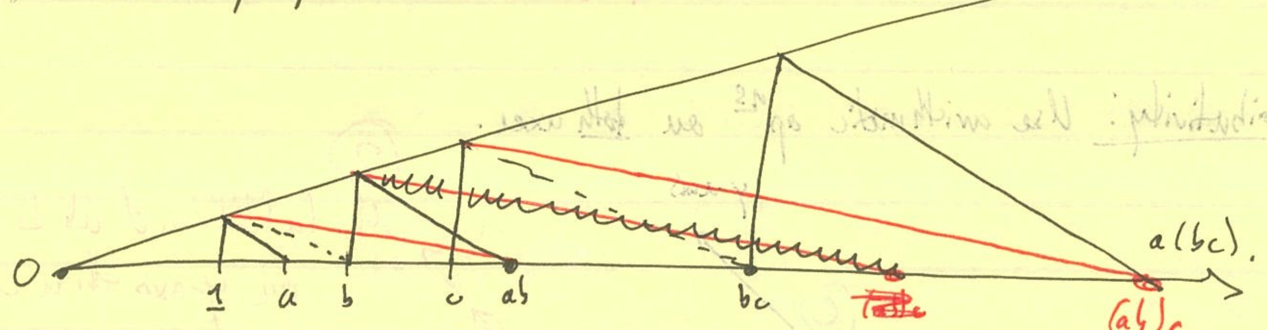


Rightmost two lines agree by scissors theorem.

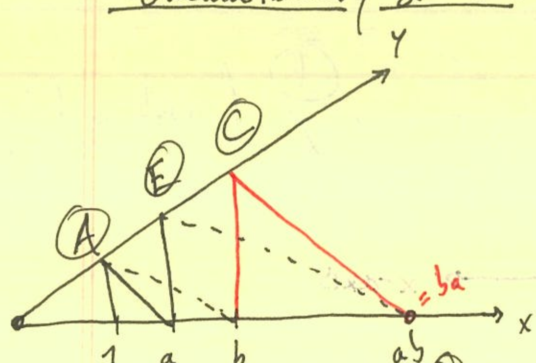
Associativity of ·:



Desargues:

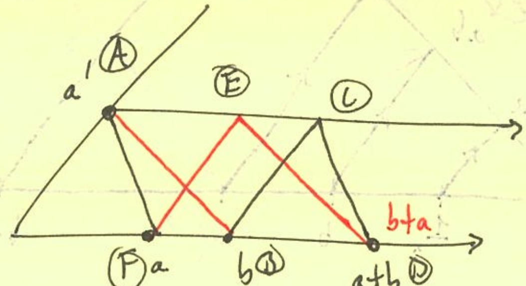


Commutativity of ·:



Pappus!

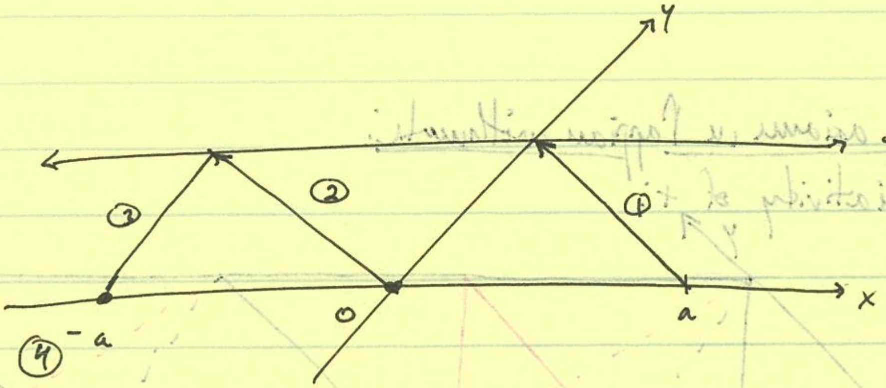
Commutativity of +:



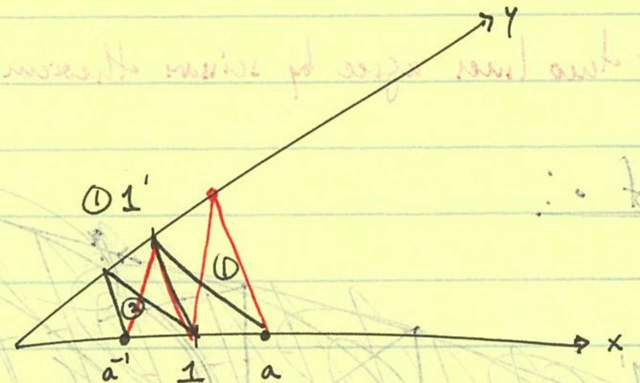
Pappus!

(In fact, little Pappus $\Leftarrow$ little Desargues.)

+ inverses:

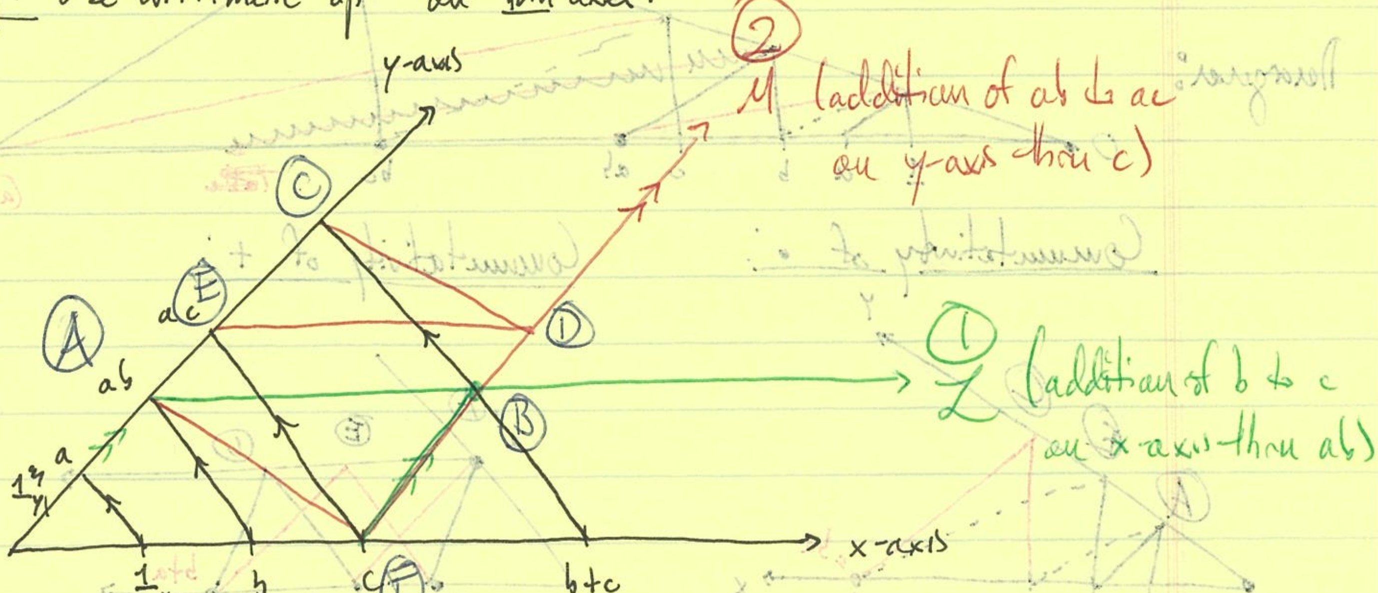


• inverses:



$1 = a\cdot a^{-1}$; construct the red lines second & use scissors.

Distributivity: Use arithmetic op$^{\text{ns}}$ on both axes.



D_4, $SO(n)$, $SO^{\underline{ns}}$.

<u>The fourth pillar</u>: <u>transformation</u> gps

We studied <u>isometries</u>: distance-preserving $f^{\underline{ns}}$ $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

- They decompose as 1, 2, or 3 reflections.
- They preserve lines, ~~+~~ circles, angles ⟵ another $\#^{\underline{ical}}$ invariant!
- They compose + have inverses.

<u>Rem</u>: There are maps $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which preserve ^inf. angle + <u>not</u> length.
These are called <u>conformal</u>, and they're important too.

Things <u>intrinsic</u> to Euclidean geom.: things invariant under transformations.
Non-eg.: "vertical." Non-e.g.: CW vs CCW. Fix this using Isom^+, which has <u>fewer</u> functions and hence <u>more</u> invariants.

It is also useful to do the other thing: <u>more</u> functions and <u>fewer invariants</u>.

<u>Ex</u>: We studied $\mathbb{R}^2$ as a vector space, w/ $u+v$ and cu.
f is <u>linear</u> when $f(u+v) = f(u) + f(v)$ and $f(c \cdot u) = c \cdot f(u)$.
f preserves <u>lines</u>, the <u>origin</u>, but not <u>distance</u> or <u>angle</u>.

PROVE ALL FOUR OF THESE.

<u>Matrices</u>: $f((x,y)) = f(x(1,0) + y(0,1)) = x f(1,0) + y f(0,1)$,
so f is determined by these two values. ~~$M = (f(1,0), f(0,1))$~~.

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix}, \quad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}. \quad M = \begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

<u>Invertibility</u>: M has an inverse when $\det M \neq 0$.
<u>Ex</u>: Scaling, rotation, reflection. $\text{Isom}^+_{\text{fixing } 0} \subseteq \text{Isom}^+ \subseteq \text{Isom} \subseteq \text{All maps}$

Even product of refl$^{\underline{ns}}$ ⊆ Inv ⊆ linear maps

$$\begin{array}{ccc} & \{id\} & \\ \text{Isom}_+ \supseteq \text{Isom}_+ \text{ fixing } O \cong SO_2 & \cap & \subseteq \\ \cap & \text{Isom fixing } O \cong O_2 & \cap \\ \text{Isom} \supseteq & & GL_2 \\ & \subseteq \text{All bij}^{\text{ns}} \supseteq & \end{array}$$

<u>Transformations of the projective spaces:</u>

Recall our model for $\mathbb{RP}^2$: $\mathbb{R}^3 \setminus \vec{0} / \mathbb{R}^\times$.

Linear maps $\mathbb{R}^3 \to \mathbb{R}^3$ carry <u>points</u> and <u>lines</u> in $\mathbb{RP}^2$ to <u>points</u> and <u>lines</u>.

What does this look like?

$$\begin{tikzcd} {[x:y:1]} \arrow[rr, mapsto, bend left] & & {[M'\binom{x}{y}+b:1]} \\ \mathbb{P}^2 \arrow[rr, "\cdot M"] & & \mathbb{P}^2 \\ \cup\!\!\mid & & \cup\!\!\mid \\ \mathbb{A}^2 & & \mathbb{A}^2 \\ (x,y) \arrow[uuuu, mapsto] \arrow[rr, mapsto] & & M'\binom{x}{y}+b \arrow[uuuu, mapsto] \end{tikzcd}$$

if we assume $M = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{M'} & b \\ \hline 0 & 0 & 1 \end{array}\right)$

- So, matrices of this form restrict to <u>affine</u> transformations of $\mathbb{A}^2$.

If M' is a product of reflections, then these give $\text{Isom}(\mathbb{R}^2)$.

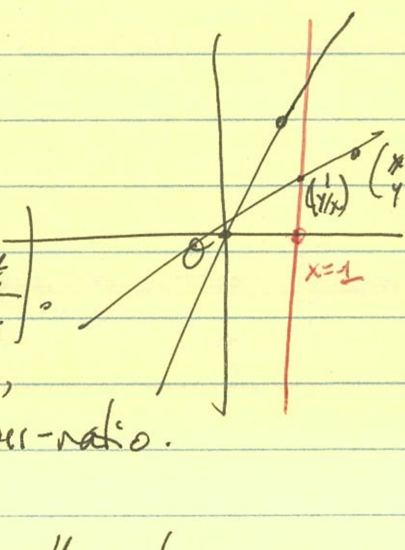
- But there are <u>many</u> <u>more</u> matrices not of this form!

For instance, <u>distance</u> is <u>not</u> an invariant.

This action is reasonably complicated. Let's look at $\mathbb{RP}^1$: $(\mathbb{R}^2 \setminus \vec{0})/\mathbb{R}^\times$.

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\quad M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$

$$M\begin{pmatrix} 1 \\ y/x \end{pmatrix} = \begin{pmatrix} a + b\frac{y}{x} \\ c + d\frac{y}{x} \end{pmatrix} \sim \begin{pmatrix} 1 \\ \frac{c+d\frac{y}{x}}{a+b\frac{y}{x}} \end{pmatrix}.$$



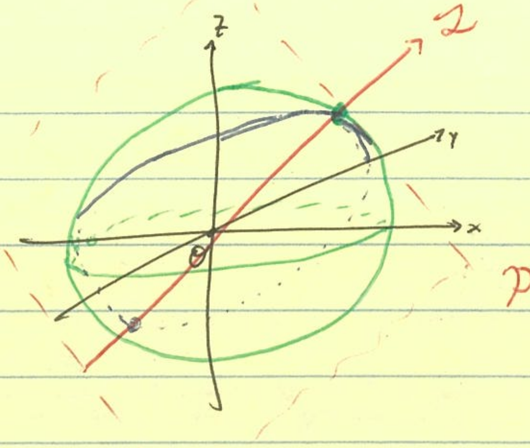
These are exactly the LFTs from before, which governed $\mathbb{RP}^1$ and preserved the cross-ratio.

<u>Ex:</u> The map $s \mapsto \frac{1}{s}$ as a matrix + in the picture.

<u>Ex:</u> There's no natural matrix rep$^{\text{n}}$ M. Can specify it up to sign.

<u>Q:</u> Is sign still an issue in $\mathbb{RP}^2$?

Spherical geometry:



The points on S^2 are $(x,y,z)\in\mathbb{R}^3$
with $|O,(x,y,z)| = 1$.
Lines are "great circles," or equators.

This has a relationship to oriented projective space, so that we can pick one of the two intersection points of L with $S(\mathbb{R}^3)$.
O

Transformation gp: $\mathrm{Isom}(\mathbb{R}^3) \supseteq$ {leaves ~~[illegible]~~ fixed} ($\supseteq$ reflections!)
($\Rightarrow$ pts of dist. 1 from O are preserved).
(It is more natural to consider self-maps $S^2 \to S^2$ which leave great circle distance fixed, as which at least have nothing to do with $\mathbb{R}^3$ more broadly. It turns out this is the same.)

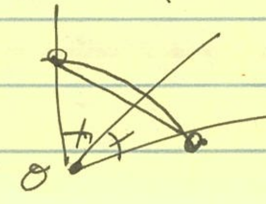
Do the 7.4 hwk

① Equidistance for straight-line metric = equidistance for great circle metric.
② The equidistant set of 2 pts in $\mathbb{R}^3$ is a plane.

STOPPED HERE

③ The equidistant set of 2 pts in S^2 is a great circle (the plane $\ni \vec{O}$).
④ The distance from 3 fixed noncollinear pts determines a 4th point.
⑤ An isometry of S^2 is determined by its behavior on 3 pts, noncollinear.
⑥ Any isometry can be modeled as 1, 2, or 3 reflections.

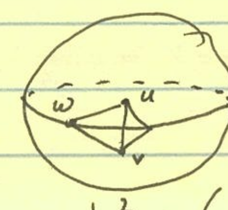
①



Angle bisectors
= ~~[illegible]~~ chord bisectors

~~$(u-w)^2$~~

②



$(u-w)^2 = (v-w)^2$
$\Rightarrow u^2 - v^2 = 2(u-v)w$.
③ " 0

④ If $P \neq Q$, then A, B, C lie on the non-degenerate equidistant line of $P + Q$.
⑤ Ditto.
⑥ Build the iso^m one refl^n at a time.

Isom$^+(S^2)$ and rotations:

~~Recall that we "defined" Isom$^+$ by disallowing single reflections. We also know that any iso^{m} of $\mathbb{R}$ decompose as product of reflections — isn't it strange that~~

There is an analogue of Isom$^+$ for S^2:

rotations $\in$ Isom$^+(S^2)$ $\qquad$ Isom$^+(\mathbb{R}^2) \ni$ rotations, translations

$\cap|$ $\qquad\qquad$ $\cap|$

rotations, refl^{ns} $\in$ Isom(S^2) $\qquad$ Isom$(\mathbb{R}^2) \ni$ rotations, translations, refl^{ns}, glide refl^{ns}

We want to prove this.

Like you did on your homework, once you know a 3 Refl^{ns} Theorem, you can classify the "types" of iso^{ms}.

To show these form subgp^s, we need to show these types are stable under composition.

Euclidean case: Translation $\circ$ translation$'$ = translation$''$.

Rotation $\circ$ trans. = trans$'$ $\circ$ Rotation.

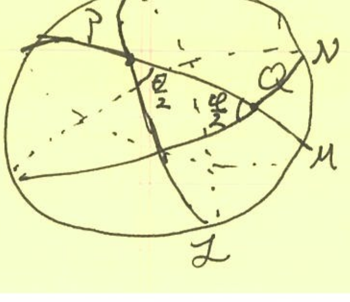
Takes some thought! $\rightsquigarrow$ Rotation $\circ$ Rotation$'$ = trans$'$ $\circ$ Rotation.

(Use $M = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{SO(2)} & \vec{t} \\ 0 & 0 & 1 \end{array}\right]$.)

Spherical case: Rotation $\circ$ Rotation$'$ = Rotation$''$ (!!).

rotation thru θ about P $\quad$ thru φ about Q.

$(r_L \circ r_M \qquad r_M \circ r_N) = r_L r_N$, a rotation. :)



Rotations of S^1:

- abelian
- identified with S^1, based on where e^0 gets sent.

Rotations of S^2:

- nonabelian
- specified by 3 #s:
  - a point on S^3, by azimuth and elevation, plus an angle of rotation
  - pitch, yaw, and roll.
- quaternions??

Quaternions + Rotations.

Remember: $\mathbb{H} = \{ a + bi + cj + dk \mid a,b,c,d \in \mathbb{R} \}$

subject to $ij = k$, $jk = i$, $ki = j$, and $i^2 = j^2 = k^2 = -1$.

We saw them before b/c $\mathbb{H}P^2$ is a Desarguesian but non-Pappian P.P.

They are also normed: $\bar{q} = a - bi - cj - dk$, $|q|^2 = q \cdot \bar{q}$,

$\overline{pq} = \bar{q} \cdot \bar{p}$, and $|pq| = |p||q|$.

$\Longrightarrow$ multiplication by q scales distances by q,

so $|q| = 1$ means $\cdot q$ is an isometry $\mathbb{H} \to \mathbb{H}$.

Idea: Use an embedding $\mathbb{R}^3 \subseteq \mathbb{H}$ and use $q\cdot$ to get isometries.

Doesn't quite work: q moves 3-space around.

Fix: Use $p \longmapsto q \cdot p \cdot q^{-1}$ instead.

Lem: This gives an iso$^{\underline{m}}$ of $\mathbb{R}^3$ fixing O.

Lem: Any such iso$^{\underline{m}}$ is induced by such a q.

Pf: Write $q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} \cdot \vec{n} \longleftarrow$ purely imaginary.

$p \longmapsto q p q^{-1}$ fixes $\vec{n}$ and rotates by θ about $\vec{n}$.

Remember: These are exactly the iso$^{\underline{ms}}$ of $S^2 \subseteq \mathbb{R}^3$ too.

In total, this gives a map

$$\begin{tikzcd} S^3 \arrow[r, "\subseteq \mathbb{H}"] \arrow[d] & \mathrm{Isom}^+(S^2) \\ \mathbb{R}P^3 \arrow[ur, "\cong"] & \end{tikzcd}$$

but 1 and -1 induce the same rotation.

So, $\mathbb{R}P^3$ has a gp str! "Accidental iso$^{\underline{m}}$".

Rem: Gimbal lock: Rep$^{\underline{n}}$ a rotation by

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \cdot \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

When $\beta = \pi/2$, this gives

$$\left(\begin{array}{l}\alpha = \text{pitch} \\ \beta = \text{yaw} = \pi/2 \\ \gamma = \text{roll} (= \text{pitch again})\end{array}\right) \begin{array}{l} X \text{ rot} \\ Y \text{ rot} \\ Z \text{ rot}\end{array}$$ Right hand rule.

$$M = \begin{bmatrix} 0 & 0 & 1 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ -\cos(\alpha+\gamma) & \sin(\alpha+\gamma) & 0 \end{bmatrix}$$ — There is a rank drop, or a loss of degree of freedom.

People prefer to use quaternions in applied math for this reason.

Lem: $\mathrm{Isom}^{+,0}\,\mathbb{R}^4 \longleftarrow S^3 \times S^3$ by

$$(p \longmapsto q \cdot p \cdot u^{-1}) \longleftarrow\!\!\!\longmapsto q, u.$$

Pf: Such an iso$^{\underline{m}}$ preserves $\cdot$, so it is linear, and $f(c^\alpha e_\alpha) = c^\alpha f(e_\alpha)$.
$f(1)^{-1} f(-)$ is an iso$^{\underline{m}}$ preserving $\mathbb{R}^3$, so it has the form $p \longmapsto u \cdot p \cdot u^{-1}$. So,

$$f(p) = (f(1)u) \cdot p \cdot u^{-1}.$$

The quaternionic-left-diff$^{\underline{ble}}$ f$^{\underline{ns}}$ are all linear.

$\mathrm{Isom}^+(\triangle) \cong S_4$.

<u>Finite subgroups in $\mathrm{Isom}^{+}(\mathbb{R}^2)$:</u>

If G is a fin. subgp, it cannot contain a translation or a glide refl^n, since these have ∞-order.

If it has a rotation, it can have only one (about a given pt), or it would have a translation.

If it has a rotation + a refl^n, the rotation pt is on the refl^n line.

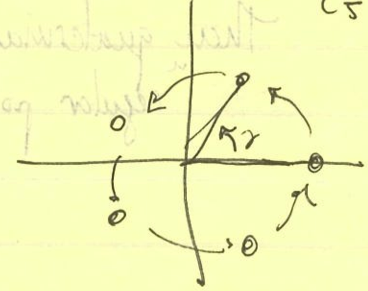
If it has more than 1 refl^n their lines all meet at a point.

$\Longrightarrow$ $G \cong$ (C_n or D_n) for some n.

~~An example in $\mathrm{Isom}^+(S^2)$:~~

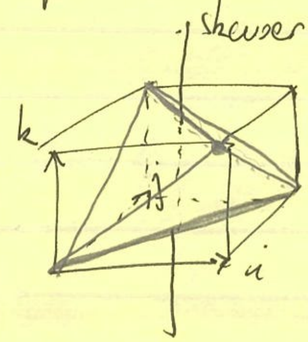
Pick a point in $\mathbb{R}^2$ and look at its orbit:

Its image forms a regular figure.



In fact, there is a correspondence between such figures + finite subgps.

An <u>example</u> in $\mathrm{Isom}^+(\mathbb{R}^3)$:



Two kinds of symmetries:

$1/3$ rotation about a vertex.

$1/2$ rotation about a skewer.

(reflection)

These are all the symmetries: 4 choices of faces to put in front, 3 choices of edge

$\rightsquigarrow$ these rotations generate a subgp of order 12. (A_4)

(+ reflection) (24) (S_4).

Thus (Klein?): $C_n, D_n, A_4, S_4, A_5 \subseteq \mathrm{Isom}^{+,0}(\mathbb{R}^3)$.

is an exhaustive list.

$$\left(q = \cos\frac{\theta}{2} + (li + mj + nk)\cdot \sin\frac{\theta}{2} \right)$$

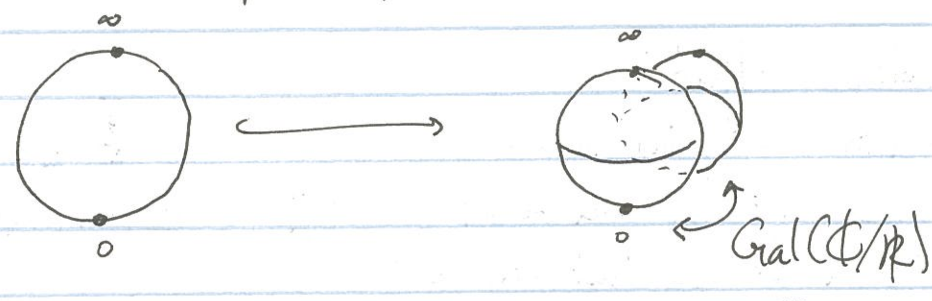
Labeling the axes by $\mathbb{H}$, the $\frac{1}{2}$ turns are encoded by
$$\pm i,\ \pm j,\ \pm k \ \left(= \cos\frac{\pi}{2} + (0i + 0j + 0k)\cdot \sin\frac{\pi}{2} \right).$$

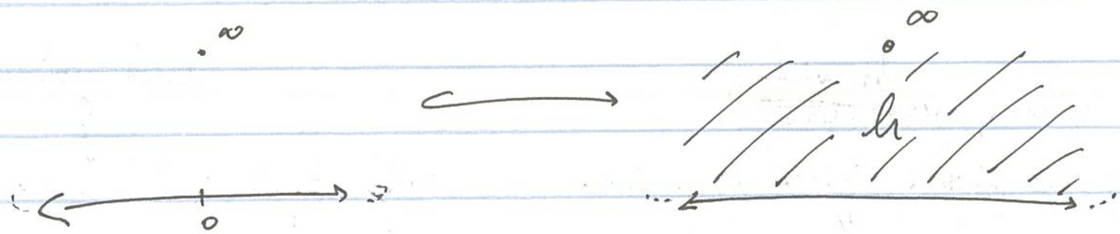
The $\frac{1}{3}$ rotations are encoded by the <u>axes</u>
$$\frac{1}{\sqrt{3}}(\pm i \pm j \pm k) \quad (\text{for all } 8 \text{ choices of signs})$$
and hence the <u>elements</u>
$$\frac{1}{2}(\pm 1 \pm i \pm j \pm k),$$
giving 24 quaternions in all.

These quaternions themselves form the vertices of a "regular polytope" $\subseteq \mathbb{H} \cong \mathbb{R}^4$, called "the 24-cell".

Introduction to Hyperbolic Space:

We will build hyperbolic space as an <u>extension</u> of $\mathbb{RP}^1$.





$\mathbb{RP}^1$ "upper half-space"

<u>Note</u>: $\mathrm{Isom}(\mathbb{RP}^1)$ is full of LFTs:

$$\frac{ax+b}{cx+d} \text{ OR } x+t,\ k\cdot x,\ \frac{1}{x} \longmapsto z+t,\ \begin{matrix} k\cdot z \text{ for } k\geq 1, \\ -\bar{z} \text{ for } k=-1 \end{matrix},\ \frac{1}{\bar{z}}.$$

<u>equations in h, need $\bar{z}$ to fix $\det<0$.</u>

These give <u>translation</u>, <u>scaling</u>, <u>reflection</u>, and <u>circle inversion</u>.

[Rem: You can build reflection through a sphere in any $\mathbb{R}^n \cup \infty$.]

[Rem: If the constants $\notin \mathbb{R}$, then move h outside itself.]

<u>Thm</u>: $\mathrm{Isom}(\mathbb{CP}^1; \mathbb{RP}^1)$ "of $\mathbb{CP}^1$ fixing $\mathbb{RP}^1$" $\cong \mathrm{Isom}(\mathbb{RP}^1)$.

You can "see more" of the behavior of LFTs here. We will show that they are <u>conformal</u>, i.e., they <u>preserve local angle</u>. This is what "preserves x-ratio" <u>really</u> means.

We will also pursue something else: $\mathfrak{h}$ can be used to give a model of a <u>non-Euclidean plane</u>. <u>All</u> of Euclid's axioms hold except the parallel postulate. (Cf. $\mathbb{RP}^2$, which had no "length.")

Euclid's atomic objects:

- Points (= points in $\mathfrak{h}$)
- Lines (= semicircles on $\mathbb{RP}^1$) ⇝ need <u>length</u>, <u>angle</u>.
- Circles (= equidistant sets)

Calculus

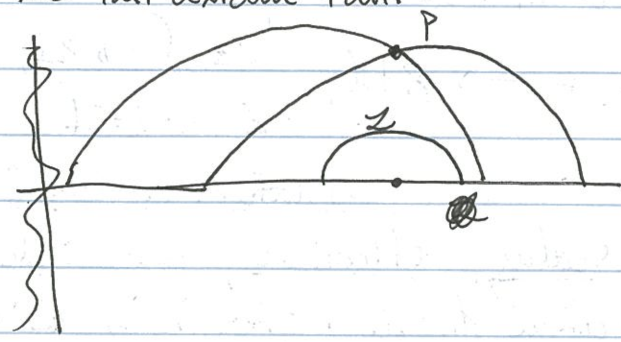
LFTs

Euclid's axioms, ~~summarized~~ abbreviated:

- Any two points are joined by a line.
- Any segment extends to a line.
- Circles exist and intersect when ①②.
- All $\llcorner$'s are "the same", where $\llcorner$ is defined by ⟋⟍.
- For $P \notin \mathcal{L}$, $\exists!\ \underbrace{\mathcal{L}' \parallel \mathcal{L}}_{\text{not meeting each other.}}$ and $P \in \mathcal{L}'$.

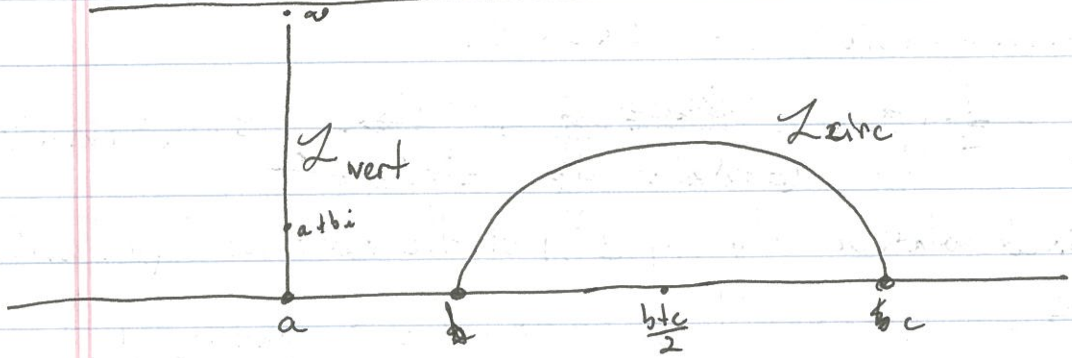
need motion / isometries

Rem: The last axiom fails:



Rem: This means we must be able to produce triangles w/ interior sum $< \pi$.

Lines and transformations



Equation for $\mathcal{L}_{vert}$: $\quad z + \bar{z} = 2a.$

Equation for $\mathcal{L}_{circ}$: $\quad \left| z - \frac{b+c}{2} \right| = \left| \frac{b-c}{2} \right|$

$$\left(z - \frac{b+c}{2}\right)\left(\bar{z} - \frac{b+c}{2}\right) = \left(\frac{b-c}{2}\right)^2$$

($\underline{Q}$: How is this related to Galois-invariance?)

$$z\bar{z} - \left(\frac{b+c}{2}\right)(z + \bar{z}) + bc = 0.$$

Claim: LFTs preserve ~~lines~~ the class of lines.

$\mathcal{L}_{vert}$ under $\frac{1}{z}$: $\frac{1}{z} + \frac{1}{\bar{z}} = 2a \rightsquigarrow \frac{1}{2a} z + \bar{z} = \frac{\cancel{2a}}{\cancel{2a}} z\bar{z}.$

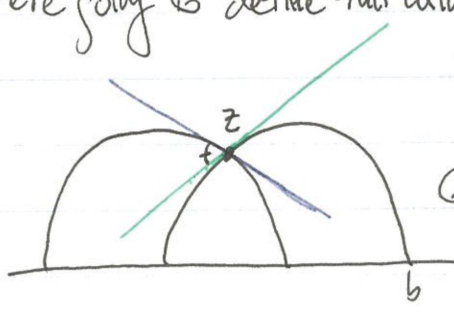
$$\rightsquigarrow z\bar{z} - \frac{1}{2}\left(\frac{1}{a} + 0\right)(z + \bar{z}) + \frac{1}{a} \cdot 0 = 0.$$

$\mathcal{L}_{circ}$ under $\frac{1}{z}$ has two cases — kind of messy.

Cor: LFTs can carry any line in $\mathfrak{h}$ to any other.

Cor: LFTs carry triangles to triangles.

Remember that we were also interested in transporting angles! Were going to define this using calculus and tangent vectors.



① It's believable that this is true for translations and dilations.

② By moving b, green + blue can intersect at any angle, so nothing special about lines.

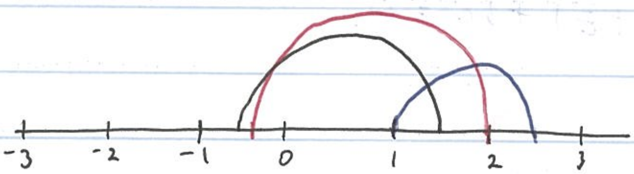
Take $\Delta z = \varepsilon(\cos\theta + i\sin\theta)$ for $0 < \varepsilon \ll 1$.

Then $\left(\frac{1}{z+\Delta z} - \frac{1}{z}\right)\frac{\cancel{z}}{\cancel{\Delta z}} = \frac{-\Delta z}{(z+\Delta z)z} \cancel{\frac{1}{z^2+\Delta z\, z}} \approx (\cos\theta + i\sin\theta)\frac{1}{z^2}$,

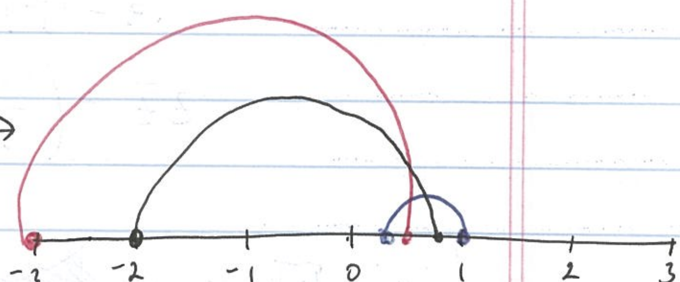
which is a fixed rotation (by $\arg(z^2)$) of the vector we began with.

$\Longrightarrow \frac{1}{z}$ preserves local angle. ($\equiv$ is <u>conformal</u>.)

Try to draw this:



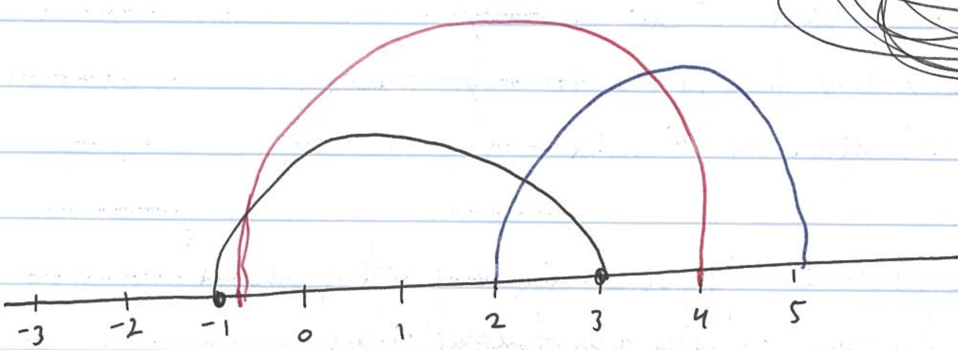
$\xrightarrow{\frac{1}{z}}$



$\Downarrow$ 2.

This is pretty unbelievable.

Mathematica drawing...?



This is a terrible drawing.

Distance in $\mathbb{H}$:

Distance in the upper half-plane $\mathbb{H}$ is extracted from the x-ratio. This is expected: isometries preserve distance, but Isom($\mathbb{C}P^1$) preserve x-ratio. These must be related.

Observations:

① x-ratio on a hyperbolic line is real.
This is true for points on $\overline{0\infty}$: $\frac{(pi-qi)(ri-si)}{(pi-ri)(qi-si)}$ cancels.

→ It's true in general, since any line is moved to another by a Möbius transformation, and these preserve or conjugate x-ratio. ← orientation-preserving isom.

② For two points qi and ri on $\overline{0\infty}$, get $R(0, q, r, \infty) = q/r$.

③ This isn't invariant under reflecting isoms, viz., $z \longmapsto \frac{1}{\bar{z}}$.
It has the effect of interchanging the ratio: r/q.

④ To get an invariant fn, we set $|qr| = |\log q/r|$.

Consequences:

Ⓐ Distance is additive: $\left|\log\frac{q}{s}\right| = \left|\log\frac{q}{r}\cdot\frac{r}{s}\right| = \left|\log\frac{q}{r} + \log\frac{r}{s}\right| = \left|\log\frac{q}{r}\right| + \left|\log\frac{r}{s}\right|$ ← order.

Ⓑ Lines extend indefinitely w/ this notion of distance:
The pts $2^n \cdot i$ are evenly spaced on $\overline{0\infty}$.

(C) For other lines, let p and s be the impact points on $\mathbb{RP}^1$.

Then $$|qr| = \left| \ln \frac{(q-p)(r-s)}{(q-s)(r-p)} \right|.$$

For $q = t + Re^{i\theta}$ and $r = t + Re^{i\varphi}$,

$$|qr| = \left| \ln \frac{\tan \theta/2}{\tan \varphi/2} \right|.$$

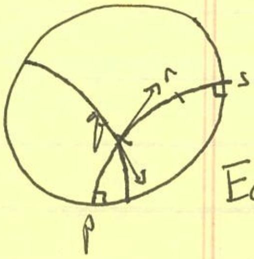
(D) SAS congruence follows.

Other models of the Hyperbolic Plane

A) $\mathbb{H}$, upper half-space, with lines vertical or semicircles on $\mathbb{R}$.

B) The Poincaré disk:

Points: $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

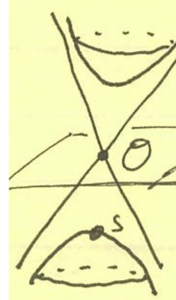


Lines: Circles or straight lines $\perp$ to the edge $\{z \in \mathbb{C} \mid |z| = 1\}$.

Distance: $|\ln(\text{cross-ratio})|$. Angle: Tangent vectors @ intersection.

Equivalence w/ $\mathbb{H}$: the LFT sending $(0, 1, -1) \longmapsto (1, 0, \infty)$.

C) The hyperboloid of 2 sheets:



Points: $H = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1 \text{ and } z > 0\}$.

Lines: Intersections w/ planes through O.

Distance: Define $\beta(\vec{u}, \vec{v}) = u_1v_1 + u_2v_2 - u_3v_3 = u^T A v$, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Then $u, v \in H \iff \beta(u,u) = -1 = \beta(v,v) \implies \beta(u,v) \le -1$.

Finally, set $|\cosh^{-1}(-b(u,v))| = d(u,v)$.

Angle: A line $L = \{ax + by + cz = 0\}$ has a pole $(a, b, -c) = \xi$ w/ $\beta(\xi,\xi) = 1$

Let M have pole η, then $\angle LM := \cos^{-1} \beta(\xi, \eta)$.

Equivalence w/ D:

Project from S. This intersects $z = 0$ somewhere in D.

D) Klein-Beltrami ~~consid~~: homogenize $x^2 + y^2 = 1$ to get...

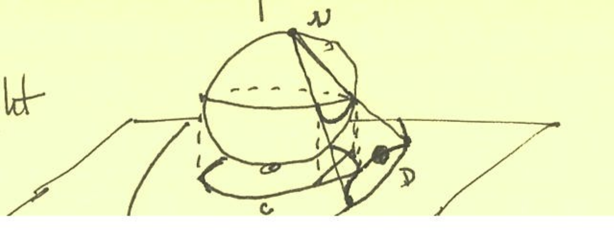
Points: $x^2 + y^2 - z^2 = 0$ in $\mathbb{RP}^2$ form the edge, C.

Lines: Projective lines (or parts of) lying inside of C.

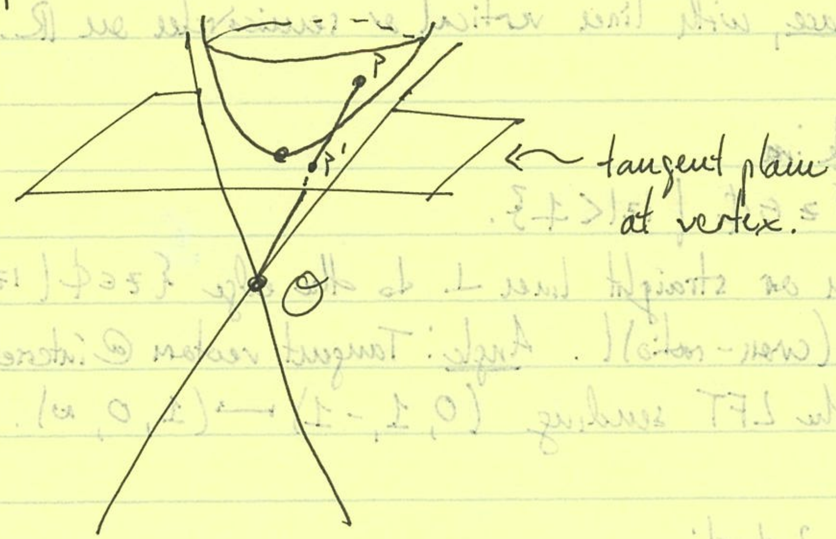
Distance: $d(q,r) = |\ln(\text{x-ratio w/ intersection pts on } C)|$.

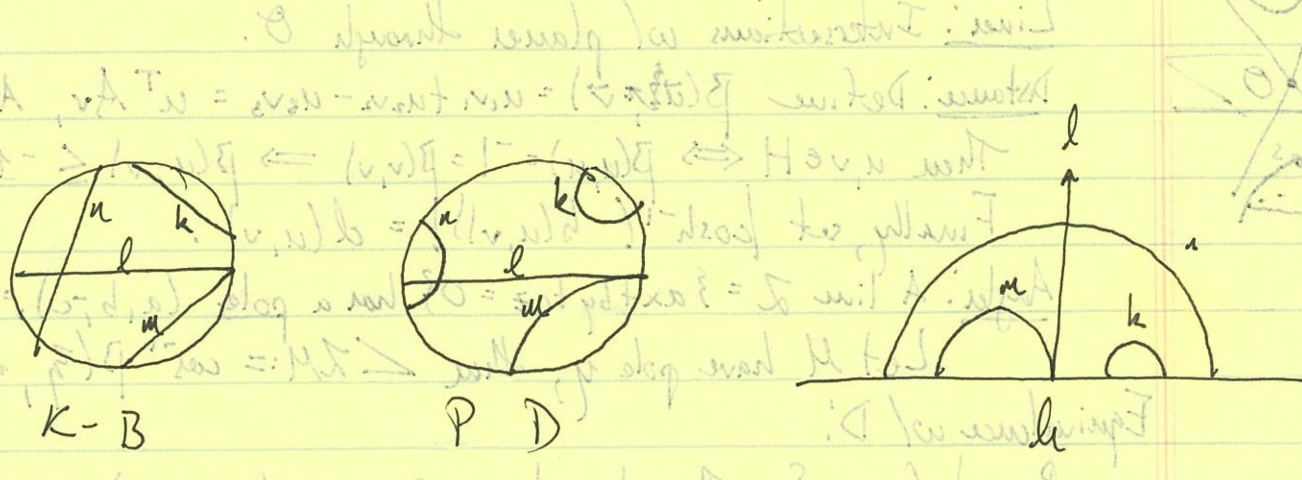
Angle: Same as in C.

Equivalence w/ D: Diagram at right shows transferring lines.



Equivalence of K-B w/ H:





K-B P D

(Types of) <u>Hyperbolic motions/isometries:</u>

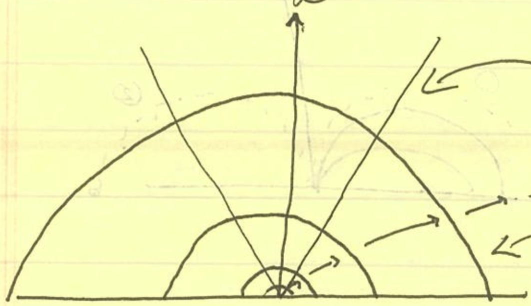
<u>Translations:</u> i.e., on a subset.

"Fix" an axis of translation, $\mathcal{L}$. No $\bar{z}$'s involved.

$\mathcal{L} = (x=s)$: translation looks like $z \mapsto k(x-s)+s$.

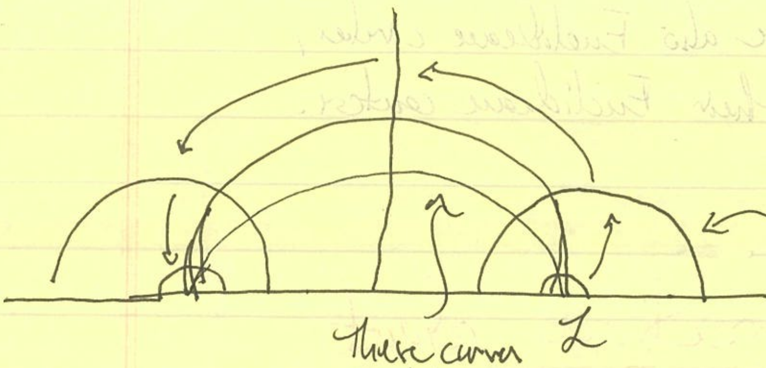
$\mathcal{L} = (|z|^2 = r)$: $z \longmapsto \frac{r(az+c)}{z+ar}$ for $a>0$.

$\rightsquigarrow \mathcal{L} = (|z-s|^2 = r)$: $z \longmapsto \frac{r(a(z-s)+c)}{(z-s)+ar} + s$ for $a>0$.



These curves are fixed b/c they're "equidistant" from $\mathcal{L}$. They are not lines.

These ~~circles~~ lines are carried into each other.



These ~~lines~~ are carried into each other.

These curves are equidistant + hence preserved.

<u>Reflections:</u> Fixes an axis of reflection <u>exactly</u>.

$\mathcal{L} = (x=s)$: $z \longmapsto -(\bar{z}-s)+s$.

$\mathcal{L} = (|z|=1)$: $z \longmapsto \frac{1}{\bar{z}}$. This "fixes" lines through reflected endpoints.

These are "uninteresting" b/c they are discrete.

Then, limit rotations.

Rotations: These are given by pairs of intersecting reflections.

Ex: $z \longmapsto \frac{1}{\bar{z}} \longmapsto \frac{-1}{z}$ is rotation by π about i.

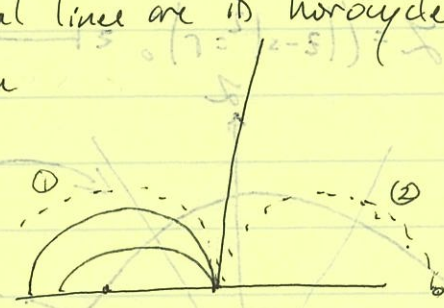
Limit rotations: The two lines of refl^{n} meet on $\bar{h}$ but not h.

Def^{n}: Each curve mapped onto itself is called a limit circle or horocycle.

Ex: $z \longmapsto z+1$ is the limit rotation of $x=0$, $x=\frac{1}{2}$ meeting at ∞. The horizontal lines are its horocycles.

Ex: $z \longmapsto \frac{z}{1-z}$ is refl^{n} in $x=0$ and $|z-1|=1$.

The horocycles are the circles tangent to the origin.



Rem: Non-Euclidean circles are also Euclidean circles, but their centers are not their Euclidean centers.

SEE: Correction about horocycles email, c. 4/18/2016.

Three Reflections in $\mathbb{H}$:

We have already proven a decomposition theorem for Isom($\mathbb{RP}^1$): everything decomposes as translations, refl^{ns}, dilations, and inversions.

Lem: There exists an iso^{m} $(p,q,r,s) \rightsquigarrow (q,p,s,r)$.

Pf: $$\frac{(r-p)(s-q)}{(r-q)(s-p)} = \frac{(s-q)(r-p)}{(s-p)(r-q)}.$$

Cor. ~~Rem~~: If f interchanges two points ($f(p)=q$ and $f(q)=p$), then f is an involution: $f \circ f = id$.

Pf: Let r be unfixed by g + set $s = g(r)$.
Then $\exists\, h : (p,q,r,s) \rightsquigarrow (q,p,s,r)$ by Lem, and it agrees with g by 3 point determination.
But then $g(g(r)) = g(s) = h(s) = r$. So, $g \circ g = id$.

Two involutions Thm: Any $f \in$ Isom($\mathbb{RP}^1$) is the $\circ$ of two involutions.

Pf: Case 1: f is the identity. Done.
Case 2: Set $r = h(p) \neq p$, $q = h(r)$. If $q = p$, use Cor.
Case 3: $q \neq p$ and $q \neq r$ by the existence of f^{-1}.
Build $g : (p,q,r) \rightsquigarrow (q,p,r)$.
Then $g \circ f : (p,q,r) \rightsquigarrow (r, \ q \ ) \longrightarrow (r, gh(q), p)$,
so $g \circ f = h$ is an involution.
Finally, $f = g^{-1} \circ h$.

g is a ref$^{\underline{n}}$ by design.

h is an involution $\Longrightarrow$ at most 2 refl$^{\underline{ns}}$.

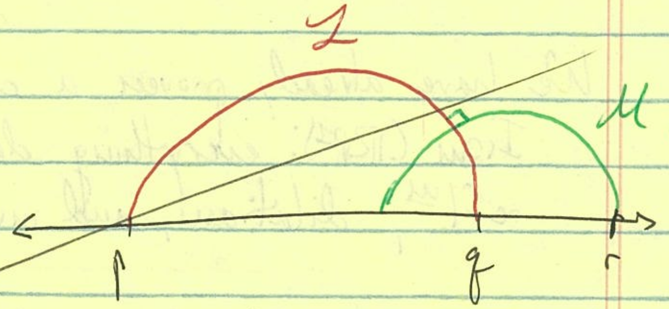
Now we tie this to h:

Consider $g : (p, q, r) \rightsquigarrow (q, p, r)$.

It preserves $\mathcal{L}$.

It preserves angle.

$\Longrightarrow$ It equals refl$^{\underline{n}}$ across $\mathcal{M}$.

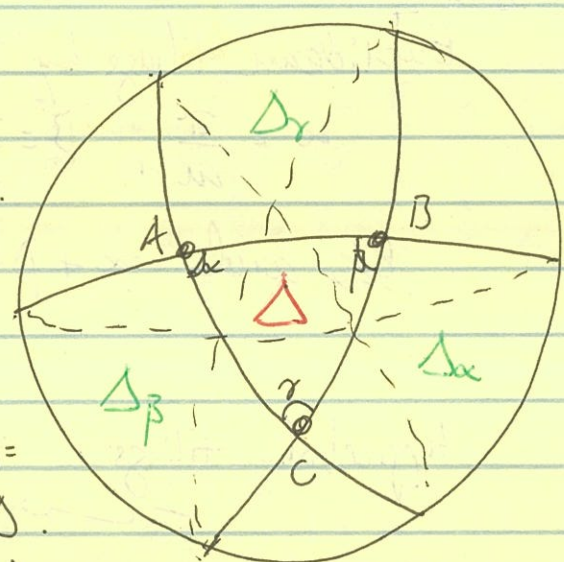


Similarly, h is modeled as a refl$^{\underline{n}}_{q}$ if it has a fixed point on $\mathbb{RP}^1$. If not, then it must at least have a fixed point on the line connecting the two points it interchanges $((p, r) \rightsquigarrow (r, p))$.

So, it preserves the $\perp$ lines$^{\mathcal{M}}$ through $\overline{pr}^{\,\mathcal{L}}$, and it must interchange its endpoints. Hence, h is ref$^{\underline{n}}$ in $\mathcal{L}$ and $\mathcal{M}$.

Area in non-Euclidean geometry

Start w/ spherical geometry:



Area of a lune: $A_\theta = A \cdot \frac{\theta}{2\pi}$.

① $2\Delta_\alpha + 2\Delta_\beta + 2\Delta_\gamma + 2\Delta = A$.

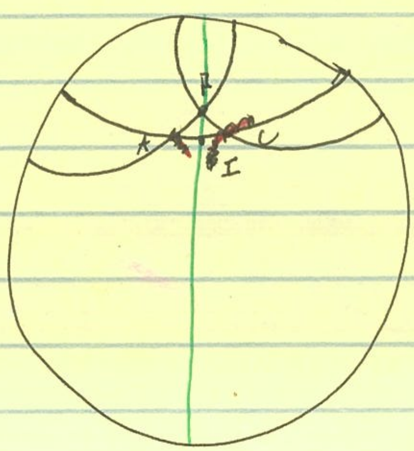
② $2(\Delta_\alpha + \Delta) + 2(\Delta_\beta + \Delta) + 2(\Delta_\gamma + \Delta) = A + 4\Delta$.

$$2\left(\frac{A}{2\pi}\cdot\alpha + \frac{A}{2\pi}\cdot\beta + \frac{A}{2\pi}\cdot\gamma\right) = A + 4\Delta$$

$$\Delta = \frac{A}{4\pi}\left(\alpha+\beta+\gamma-\pi\right).$$ For $A = 4\pi$, this gives $\Delta = \alpha+\beta+\gamma-\pi$

There are various senses in which the hyperbolic plane is like a sphere of radius "i". One is that $\Delta = \pi - (\alpha+\beta+\gamma)$.

This satisfies the things you would expect of area:

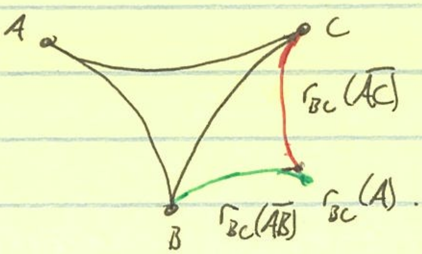


$$\Delta = \pi - (\alpha+\beta+\gamma)$$
$$\stackrel{?}{=} (\pi - (\alpha+\beta_1+\gamma_1)) + (\pi - (\beta_2+\gamma_2+\gamma)).$$

✓

There are analogues of the laws of sines and cosines for hyperbolic Δ's.

Tilings:



Area does not match Euclidean area. Appears to change as a tiling is built.

Incidentally:

Euclidean tilings by triangles:

$$\alpha = \frac{\pi}{m}, \quad \beta = \frac{\pi}{n}, \quad \gamma = \frac{\pi}{2} \rightsquigarrow (\alpha, \beta) \in \left\{ \begin{array}{l} (4,4), \\ (3,6), \ (6,3) \end{array} \right\}.$$

and $\alpha + \beta + \gamma = \pi$.

Hyperbolic tilings: $\alpha + \beta + \gamma < \pi$.

$\rightsquigarrow$ ∞^{ly} many sol$^{\text{ns}}$.

Families of geometries:

Def^n: Remember that a point in $\mathbb{RP}^2$ is specified by a vector $\vec{p} \in \mathbb{R}^3 \setminus \vec{0}$,
a plane in $\mathbb{RP}^2$ ——//—— a covector $u \in (\mathbb{R}^3 \setminus \vec{0})^*$.

A matrix A gives rise to a correlation sending points to lines by $A: \vec{p} \longmapsto (s \cdot A\vec{p})^t$ and lines to points by $A: u \longmapsto (s \cdot uA^{-1})^*$ for a normalizing value s.

Lem: These assignments are inverses when $A = A^t$. This condition on A is called a polarity.

Def^n: Two points are conjugate when one lies on the other's polar.
Two lines are conjugate when one lies on the other's pole.

Ex: $A = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$, conjugate lines have perpendicular covectors.

Ex: Self-conjugate points form a conic:

$$0 = p^t A p = p_1^2 A_{11} + p_2^2 A_{22} + p_3^2 A_{33} + 2p_1p_2A_{12} + 2p_1p_3A_{13} + 2p_2p_3A_{23}.$$

Lem: Up to projective transformation, any A can be rewritten to have a conic of the form $p_1^2 + p_2^2 \pm p_3^2 = 0$.
$c(p_1^2 + p_2^2) + p_3^2 = 0$, $c \in \mathbb{R} \setminus 0$.

Rem: This gives a family of conics.

Construction: Associated to a conic, we form a geometry w/...
- Points = points of $\mathbb{RP}^2$ (interior to the conic).
- Lines = lines of $\mathbb{RP}^2$ (interior to the conic).
- Perpendicular = conjugate w/r/t the conic.
- Distance = $k \cdot |\ln R(A, B, P, Q)|$, where $P+Q$ are the $\cap$ of AB w/ $\mathcal{C}$
- Angle = $k' \cdot \ln R(\mathcal{L}, \mathcal{M}, \tau_1, \tau_2)$, where τ_1, τ_2 are tangents to $\mathcal{C}$ at impact pts of $\mathcal{L}$ and $\mathcal{M}$.

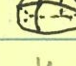
$I_{\mathcal{C}}$ = subgp of $I_{\text{proj}}(\mathbb{RP}^2)$ fixing $\mathcal{C}$.

The case $c<0$: Hyperbolic geometry
(and the scaled Klein–Beltrami model).

note the heavy reliance on algebra.

The case $c>0$: Modified elliptic geometry

$\neg$ (2 pts lie on a unique line) $\rightsquigarrow$  , spherical geom.

$\neg$ (a line separates the plane) $\rightsquigarrow$  , $\frac{1}{2}$ spherical geom.

$\simeq$ via the doubling map on the boundary.

The case $c=0$: <u>Affine</u> geometry.

Angle and distance become hard to define b/c
"$x_3=0$" defines a <u>line</u> rather than a <u>conic</u>.

Still have $\parallel$ lines as those intersecting @ an ideal point.

<u>Similarity geometry</u>: incorporate the coordinate swap $[x:y:0]\mapsto[-y:x:0]$.

Two lines are $\perp$ if their ideal pts agree under this involution.

Can actually use this to get angles generally.

<u>Euclidean geometry</u>: Fix a line segment to use as a ruler,
restrict to <u>isometries</u>.

① Isometries ↻ Euclidean geom.

- ↳ pts, lines, circles
- ↳ arithmetic of lengths, angles, areas
- ↳ rigid motions + congruence, scaling + similarity.

{ weakening axioms!

② Models. — A surplus of things, some of them relevant.

$\mathbb{R}\times\mathbb{R}$, w/ distance.

$\mathbb{R}\times\mathbb{R}$ w/ vector space str, + dot product.

③ Isometries ↻ Projective geom.

- ↳ the correspondence w/ fields ⟿ $\mathbb{RP}^2$.
- ↳ the x-ratio and LFTs.

$k P^2$ for other fields k.

and worse: Moulton, ...

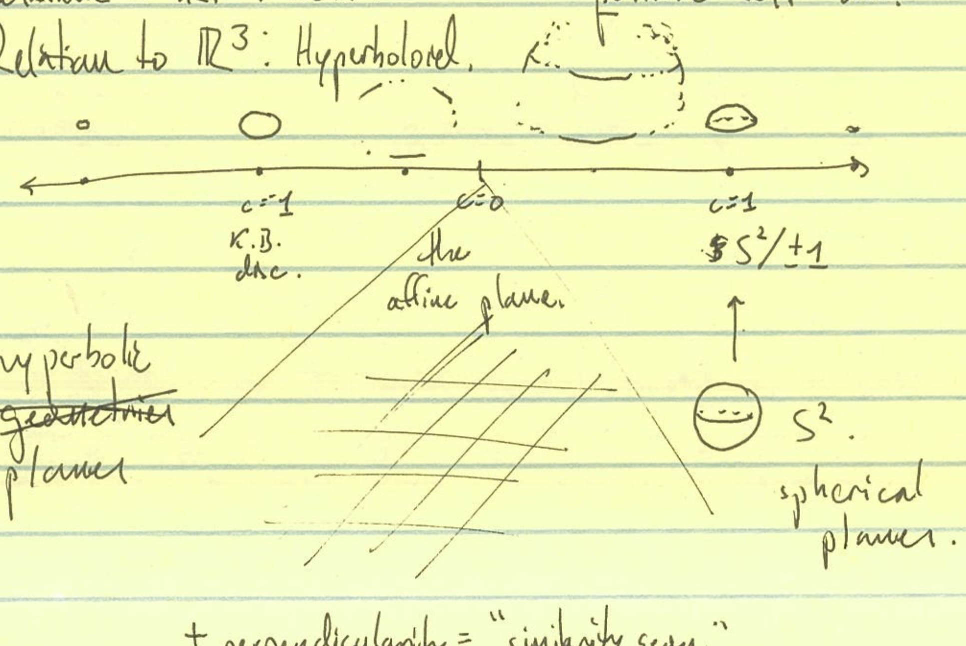
{ Attempt to reincorporate length, angle, area using a bilinear form / conic.

Hyperbolic geom.

- ↳ Isom: h and P.D.
- ↳ Relation to $\mathbb{RP}^2$: K.B.
- ↳ Relation to $\mathbb{R}^3$: Hyperboloid.

restrict attn to pts w/ positive self-dot.

$-(x^2+y^2)+z^2=0$



+ perpendicularity = "similarity geom."

+ a unit length = "Euclidean geom."