Homework #4

Math 25b

Due: March 8th, 2017

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one, a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)[1]

Throughout, $A \subseteq \mathbb{R}^n$ will be a compact rectangle.[2]

1 For submission to Thayer Anderson

Problem 1.1. This problem shows that it is possible, if tedious, to perform integration of simple functions straight from the definitions. Define $f\colon [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x + y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Without invoking Fubini's theorem, show that f is integrable and that $\int_A f = 1/2$.

Problem 1.2.

1. Let $f, g\colon A \to \mathbb{R}$ be integrable functions which satisfy $f \leq g$. Show that this transfers to their integrals: $\int_A f \leq \int_A g$.
2. Let $f\colon A \to \mathbb{R}$ be integrable. Show that $|f|$ is also integrable, and conclude $\left|\int_A f\right| \leq \int_A |f|$.

Problem 1.3. If $f\colon A \to \mathbb{R}$ is nonnegative and integrable with $\int_A f = 0$, show that any any $\varepsilon > 0$ the set $\{x \in A \mid f(x) > \varepsilon\}$ has measure zero.

[1] This version of the homework dates from March 6, 2017.

[2] You might also like to be reminded that integrable functions are defined to be bounded.

2 For submission to Davis Lazowski

Problem 2.1. If $C \subseteq A$ is a bounded set of measure zero and with integrable characteristic function χ_C, show that the integral $\int_A \chi_C$ is necessarily zero.

Problem 2.2.
1. Show that the collection of all rectangles $[a_1, b_1] \times \cdots \times [a_n, b_n]$ with all a_j and b_j rational (and n fixed) can be arranged into a sequence.
2. Conclude that if $\mathcal{O}$ is an open cover of *any* set $A \subseteq \mathbb{R}^n$, then there is a sequence of opens $U_1, U_2, \ldots \in \mathcal{O}$ chosen from the cover such that $\{U_1, U_2, \ldots\}$ forms a cover of A.[3] (Hint: show that any $U \in \mathcal{O}$ has a rational rectangle in it.)

Problem 2.3.
1. Show that an unbounded set C cannot have content zero.
2. Use this observation to give an example of a closed set of measure zero that is not of content zero.
3. If C is a set of content zero, show that its boundary ∂C has content zero.
4. Exhibit an example of a bounded set C of measure zero such that ∂C does not have content zero.

3 For submission to Handong Park

Problem 3.1. Let $f\colon A \to \mathbb{R}$ be integrable, and suppose that $g\colon A \to \mathbb{R}$ agrees with f except at finitely many points. Show that g is also integrable (with the same integral).

Problem 3.2. Let $f, g\colon A \to \mathbb{R}$ both be integrable functions.

1. For any partition P of A and for any subrectangle S of P, show
$$m_S(f) + m_S(g) \leq m_S(f+g), \qquad M_S(f+g) \leq M_S(f) + M_S(g).$$
2. Conclude that $f + g$ is integrable and that $\int_A (f+g) = \int_A f + \int_A g$.
3. For any constant $c \in \mathbb{R}$, show $\int_A cf = c \int_A f$.

Problem 3.3. Show that if $f, g\colon A \to \mathbb{R}$ are integrable, then so is their product $f \cdot g$.

4 For submission to Rohil Prasad

Problem 4.1. Show that an increasing function $f\colon [a,b] \to \mathbb{R}$ is integrable. (Hint: consider how just how little such a function can fail to be continuous.)

Problem 4.2. If A is a Jordan-measurable set and $\varepsilon > 0$, show that there is a compact Jordan-measureable set $C \subseteq A$ such that $\int_{A \setminus C} 1 < \varepsilon$.

Problem 4.3. Let $A \subseteq \mathbb{R}^n$ be a closed rectangle. Show that a subset $C \subseteq A$ is Jordan-measurable if and only if for every $\varepsilon > 0$ there exists a partition P of A satisfying
$$\sum_{S \text{ of type I}} \mathrm{vol}(S) - \sum_{S \text{ of type II}} \mathrm{vol}(S) < \varepsilon,$$
where "type I" are those rectangles intersecting C and "type II" are those rectangles contained in C.

[3] This condition is sometimes called *second-countability*. Naming schemes in general topology leave a lot to be desired; see also $T_{2\frac{1}{2}}$ *spaces*.