Practice Math 25b Midterm #2.1 Solutions

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Problem 1. Compute the derivative of the function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ given by the formula

$$f(v) = \|v\|^2 \cdot v.$$

Solution. There are at least two ways to solve this problem.

1. You can proceed directly from the definition and guess the linear part:

$$\begin{aligned} f(v+h) - f(v) &= \|v+h\|^2(v+h) - \|v\|^2 v \\ &= (\|v\|^2 + \|h\|^2 + 2\langle v, h\rangle)(v+h) - \|v\|^2 v \\ &= \underbrace{2\langle v, h\rangle v + \|v\|^2 h}_{\text{linear part}} + \|h\|^2 v + \|h\|^2 h + 2\langle v, h\rangle h. \end{aligned}$$

   So, we set $(D_v f)(h) = 2\langle v, h\rangle v + \|v\|^2 h$ and hope for the best:

$$\lim_{h\to 0} \frac{f(v+h) - (f(v) + (D_v f)h}{\|h\|} = \lim_{h \to 0} \frac{\|h\|^2 v + \|h\|^2 h + 2\langle v, h\rangle h}{\|h\|}.$$

   Indeed, each of these terms goes to zero as h tends to zero.

2. We can think of this function as the composite of the identity map $i\colon v \mapsto v$, the square-norm map $s\colon v \mapsto \|v\|^2$, and the product map $p\colon (c, w) = c \cdot w$ sending a pair of a scalar and a vector to their product. The derivative of i is the identity, and the derivative of s is

$$D_v s = \begin{pmatrix} 2|v_1| & \cdots & 2|v_2| \end{pmatrix}.$$

   The map p is bilinear, so we employ its product rule as follows:

$$\begin{aligned} D_v f &= D_{\|v\|^2, v} p \circ D_v(s \times i) \\ &= D_{\|v\|^2, v} p \circ (D_v s \times \mathrm{id}) \\ &= v \cdot D_v s + \|v\|^2 \cdot \mathrm{id}. \end{aligned} \tag{ECP}$$

Problem 2. Let $S_1, S_2, \dots$ be a countable collection of subsets of $\mathbb{R}^n$ such that S_i has content zero for every i. Set $S = \bigcup_{i=1}^{\infty} S_i$ to be their union. Prove or disprove (i.e., provide a counterexample) the following statements:

1. S has measure zero.
2. S has content zero.

Solution.

1. We will show for any $\varepsilon > 0$ that there is a countable set of rectangles $\{R_i\}_{i=1}^{\infty}$ of total volume less than ε such that $S \subseteq \cup_{i=1}^{\infty} R_i$.

   Since each S_j has content zero, we can construct for any j a set of n_j rectangles $\{R_i^j\}_{i=1}^{n_j}$ of total volume less than $\varepsilon/2^j$.

   Then, the union of all the R_i^j is a set of rectangles of volume less than $\sum_{j=1}^{\infty} \varepsilon/2^j = \varepsilon$ that covers S as desired.

2. We provide a counterexample where S does not have content zero.

   Let $\mathbb{Q}^n \subset \mathbb{R}^n$ be the set of elements with rational coordinates. Since it is countable, we can label its elements as $q_1, q_2, \dots$ and then set $S_i = \{q_i\}$.

   It is clear that all of the S_i are content zero, but their union S is equal to $\mathbb{Q}^n$. This set is unbounded but the union of a finite number of open rectangles is bounded, so clearly S cannot have content zero in this case. (RP)

Problem 3. Let $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a positive definite quadratic form.

1. Let $S = \{v \in \mathbb{R}^n : \|v\| = 1\}$ be the sphere of unit vectors in $\mathbb{R}^n$. Show that there is a $v_0 \in S$ such that for all $v \in S$,

$$Q(v) \geq Q(v_0) > 0.$$

2. Use the formula $Q(x) = \|x\|^2 Q(x/\|x\|)$ to conclude that there exists a constant $C > 0$ such that $Q(x) \geq C\|x\|^2$.

Solution.

1. S is a compact connected set, hence has compact connected image in $\mathbb{R}$ under the continuous map Q, hence is a closed interval. Since Q is positive definite, this closed interval is a subset of $(0, \infty)$, so its minimum is a positive number, achieved by some $v_0 \in S$.

2. Just chain them:

$$Q(x) = \|x\|^2 \cdot Q\left(\frac{x}{\|x\|}\right) \geq \|x\|^2 \cdot Q(v_0).$$

(ECP)

Problem 4. Find the critical points of $f(x, y) = (x^2 + y^2)e^{x^2 - y^2}$ and classify them each as a local minimum, a local maximum, or a saddle point.

Solution. This problem is really computational. Goodness. Sorry.

We compute

$$\begin{aligned} D_{x,y}f &= \begin{pmatrix} 2xe^{x^2-y^2} + (x^2+y^2)2xe^{x^2-y^2} & 2ye^{x^2-y^2} + (x^2+y^2)(-2y)e^{x^2-y^2} \end{pmatrix} \\ &= \begin{pmatrix} (x^2+y^2+1)2xe^{x^2-y^2} & (x^2+y^2-1)(-2y)e^{x^2-y^2} \end{pmatrix}. \end{aligned}$$

We are looking for values $c = (a,b)$ for which $D_c f = 0$. The first entry vanishes if and only if $a = 0$, in which case the second component specializes to

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(a,b)} = (y^2-1)(-2y)e^{-y^2}.$$

It follows that $b = 0$, $b = 1$, or $b = -1$. In each case, we will want to have access to the matrix of second partials:

$$H_{x,y}f = \begin{pmatrix} 2e^{x^2-y^2}(1+2x^4+y^2+x^2(5+2y^2)) & -4e^{x^2-y^2}xy(x^2+y^2) \\ -4e^{x^2-y^2}xy(x^2+y^2) & 2e^{x^2-y^2}(1-5y^2+2y^4+x^2(-1+2y^2)) \end{pmatrix}.$$

- Set $a = 0$ and $b = 0$. The matrix then becomes $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, hence the corresponding quadratic form is

  $$P^2_{f@(0,0)}(0+h, 0+k) - P^1_{f@(0,0)}(0+h,0+k) = \frac{1}{2}(2h^2+2k^2) = h^2+k^2.$$

  This represents a local minimum.

- Set $a = 0$ and $b = -1$. The matrix then becomes $\begin{pmatrix} 4e^{-1} & 0 \\ 0 & -2e^{-1} \end{pmatrix}$, hence the corresponding quadratic form is

  $$P^2_{f@(0,-1)}(0+h,-1+k) - P^1_{f@(0,-1)}(0+h,-1+k) = e^{-1}(2h^2-k^2).$$

  This represents a saddle point.

- Set $a = 0$ and $b = 1$. The matrix then becomes $\begin{pmatrix} 4e^{-1} & 0 \\ 0 & -4e^{-1} \end{pmatrix}$, hence the corresponding quadratic form again is

  $$P^2_{f@(0,1)}(0+h,1+k) - P^1_{f@(0,1)}(0+h,1+k) = e^{-1}(2h^2-k^2)$$

  This again represents a saddle point. (ECP)

Problem 5. Let F be the real vector space of bounded integrable functions $[0,1] \to \mathbb{R}$, let $Z \subset F$ be the subspace of integrable functions that are nonzero exactly on a subset of $[0,1]$ of measure 0, and set $L = F/Z$. Define the function

$$\begin{aligned} S\colon\ & L \to \mathbb{R}, \\ & f \mapsto \left(\int_0^1 |f|^2\right)^{1/2}. \end{aligned}$$

(Note that adding a function in Z to f does not change the integral.)

1. Show that $S(\lambda \cdot f) = |\lambda| \cdot S(f)$.

2. Show that $S(f+g) \leq S(f) + S(g)$ (Hint: Apply Cauchy-Schwarz to a well chosen inner product).

Solution. 1. This is a direct calculation:

$$S(\lambda \cdot f) = \left(\int_0^1 |\lambda f|^2 \right)^{1/2} = \left(\int_0^1 |\lambda|^2 |f|^2 \right)^{1/2} = |\lambda| \left(\int_0^1 |f|^2 \right) = |\lambda| \cdot S(f).$$

2. Squaring both sides, the inequality is equivalent to

$$\int_0^1 (|f| + |g|)^2 \leq \int_0^1 (|f|^2 + |g|^2) + 2 \left(\int_0^1 |f|^2 \int_0^1 |g|^2 \right)^{1/2}.$$

Simplifying this, we must show

$$\int_0^1 |fg| \leq \left(\int_0^1 |f|^2 \int_0^1 |g|^2 \right)^{1/2}.$$

Now, note that the function $\langle f, g \rangle = \int_0^1 fg$ is in fact an inner product on L! It is clearly bilinear. Furthermore, it is positive-semidefinite since $\int_0^1 f^2 = 0$ implies $f = 0$ except on a subset of measure zero, which is the zero element in L—this is why we passed to the quotient.

Now the inequality above is exactly the Cauchy-Schwarz inequality

$$\langle |f|, |g| \rangle \leq \sqrt{\langle |f|, |f| \rangle \cdot \langle |g|, |g| \rangle}. \tag{RP}$$

Practice Math 25b Midterm #2.2 Solutions

Eric Peterson

Problem 1. Let $U \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ denote the set of *invertible* operators. Our goal is to show that $\chi(A) = A^{-1}$ defines a differentiable function $\chi\colon U \to U$.

1. Show that U forms an open subset. (Hint: *don't* invoke epsilonics. Think of a clever continuous map and a clever open you can take the preimage of.)
2. Use the identity $A \cdot A^{-1} = I$ and the chain rule to calculate $D_A\chi$.

Solution. 1. The map $\det\colon \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$ does the job: the preimage of the open set $\mathbb{R} \setminus \{0\}$ is exactly the set of invertible operators.

2. The derivative of the right-hand side of the functional equation is zero. We access the derivative of the left-hand side using the chain rule:

$$\begin{aligned} D_A(p \circ (\operatorname{id} \times \chi))(H) &= D_{A,A^{-1}}p \circ (D_A \operatorname{id} \times D_A\chi)(H) \\ &= A \cdot D_A\chi + D_A \operatorname{id}(H) \cdot A^{-1}. \end{aligned}$$

Combining these equations and solving for $D_A\chi$, we get

$$D_A\chi = A^{-1} \cdot -H \cdot A^{-1}. \qquad \text{(ECP)}$$

Problem 2. If A is a measure zero subset of a bounded rectangle R such that the integral $\int_R \chi_A$ exists, show that the integral must be equal to zero.

Solution. As in the definition of measure zero, let $\mathcal{O}$ be an ε–cover of A by rectangles, and extend the edges of $U \in \mathcal{O}$ to form a partition P of R. Since each rectangle in the upper sum either does not intersect A or is contained in one of the opens $U \in \mathcal{O}$, we have $\mathcal{U}_f(P) \leq \varepsilon$. (ECP)

Problem 3. We say a sequence of functions $(f_n)\colon \mathbb{R} \to \mathbb{R}$ converge pointwise to $f\colon \mathbb{R} \to \mathbb{R}$ if for every $x \in \mathbb{R}$, the sequence $(f_n(x))$ converges to $f(x)$.

1. Exhibit a sequence (f_n) of continuous functions that converge pointwise to a non-continuous function f.

2. Exhibit a sequence (f_n) of continuously differentiable functions that converge pointwise to a function f that is continuous but not differentiable everywhere. (Hint: Consider $f(x) = |x|$.)

Solution. 1. A classic example is $f_n(x) = x^n$ on $[0,1]$, converging pointwise to

$$f(x) = \begin{cases} 0 & \text{if } x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

2. You can guess an example as in the hint as follows: a rational function has limiting behavior given by its leading terms, so consider something like

$$|x| = \sqrt{x^2} = \sqrt{x^{n+2}/x^n} \approx \sqrt{\frac{x^{n+2}}{x^n + 1}} \text{ near } \infty.$$

This has good behavior for $x > 1$, but poor behavior for $0 \le x < 1$. This window can be narrowed by introducing an extra factor:

$$|x| \approx \sqrt{\frac{(nx)^{n+2}}{n^{n+2}x^n + 1}}.$$

This now has good behavior on $x > 1/n$ and hence the sequence converges pointwise to $|x|$. To see that f_n is differentiable, the chain rule mostly suffices, except at 0 where we make the manual check

$$\lim_{x\to 0} \frac{\sqrt{\frac{(nx)^{n+2}}{n^{n+2}x^n+1}}}{x} = \lim_{x\to 0} \sqrt{\frac{(nx)^{n+2}}{n^{n+2}x^{n+2} + x^2}} = \lim_{x\to 0} \sqrt{\frac{1}{1 + n^{-(n+2)}x^{2-n}}}.$$

For $n > 2$, the denominator tends to $\pm\infty$ and hence the fraction tends to 0, showing f_n differentiable at 0 with derivative 0. (ECP)

Problem 4. 1. Recall that the Cantor set is constructed by removing the middle one-third open interval $(1/3, 2/3)$ from the unit interval, then removing the middle one-third open intervals from the two intervals that remain, and so on, at each step removing the middle one-third open intervals. Show that the Cantor set is of measure zero.

2. Now perform the Cantor set construction, but at the n^{th} step remove the middle strip of each interval of size $1/4^n$ (not considered as a percentage of the volume of the subinterval). This is called a *fat Cantor set*. Calculate its volume.

Solution. 1. Each of the stages of the Cantor set construction give a cover of the finished Cantor set by rectangles of discernable volume. At the n^{th} stage, there are 2^n remaining rectangles, all disjoint and each of volume $1/3^n$. Hence, their total volume is $2^n/3^n$, which tends to 0 as n tends to ∞.

2. We phrase the above calculation slightly differently to accommodate the non-proportional setup. The initial fat Cantor set approximation is a single rectangle of volume $V_0 = 1$. When forming the n^{th} approximation from the $(n-1)^{\text{st}}$, we remove rectangles of volume $1/4^n$ from 2^{n-1} many intervals, so that $V_n = V_{n-1} - 2^{n-1}/4^n$. We therefore calculate:

$$V_\infty = 1 - \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2^n}. = 1 - \frac{1}{2} = \frac{1}{2}. \tag{ECP}$$

Problem 5. Let $B_R^n \subseteq \mathbb{R}^n$ denote the ball of radius R, centered at the origin.

1. Justify the equation $\operatorname{vol}(B_R^n) = R^n \operatorname{vol}(B_1^n)$.
2. Use Fubini's theorem to justify the equation

$$\operatorname{vol}(B_1^n) = \operatorname{vol}(B_1^{n-1}) \cdot \int_{-1}^{1} (1 - x_n^2)^{\frac{n-1}{2}} \, \mathrm{d}x_n.$$

3. Find a recursive expression for the sequence $c_n = \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}} \, \mathrm{d}t$ as n varies.
4. Conclude that the ratio of the volume of the unit ball to the volume of the cube that circumscribes it tends to zero as n tends to ∞.

Solution. 1. B_R^n is the rescaling of B_1^n by a factor of R, and volume scales by the n^{th} power in $\mathbb{R}^n$. If you like, you can justify this with a change-of-coordinates formula: the map $z \mapsto Rz$ maps B_1^n to B_R^n, and the Jacobian of $z \mapsto Rz$ is R^n.

2. As instructed, we apply Fubini's theorem:

$$\begin{aligned}
\operatorname{vol}(B_1^n) &= \int_{B_1^n} 1 = \int_{[-1,1]^{\times n}} \chi_{B_1^n} \\
&= \int_{-1}^{1} \left(\int_{[-1,1]^{\times n}} \chi_{B^{n-1}_{\sqrt{1-x_n^2}}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} \right) \mathrm{d}x_n \\
&= \int_{-1}^{1} \sqrt{1-x_n^2}^{(n-1)} \cdot \operatorname{vol}(B_1^{n-1}) \, \mathrm{d}x_n \\
&= \operatorname{vol}(B_1^{n-1}) \cdot \int_{-1}^{1} (1-x_n^2)^{\frac{n-1}{2}} \, \mathrm{d}x_n.
\end{aligned}$$

3. This is a fun exercise in classical calculus. The quadratic in our integrand is hiding under a quadratic radical, so we begin by making the coordinate transformation $t = \sin\theta$:

$$c_n = \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}} \, \mathrm{d}t = \int_{-\pi/2}^{\pi/2} \cos^n \theta \, \mathrm{d}\theta.$$

Then, we split the integrand up into $\cos^n\theta\,\mathrm{d}\theta = \cos^{n-1}\theta\cdot\cos\theta\,\mathrm{d}\theta$ and apply integration by parts to get

$$c_n = \int_{-\pi/2}^{\pi/2}\cos^n\theta\,\mathrm{d}\theta = \left[\cos^{n-1}\theta\cdot\sin\theta - \int\cos^{n-2}\theta\cdot\sin\theta\cdot(n-1)\cdot(-\sin\theta)\,\mathrm{d}\theta\right]_{-\pi/2}^{\pi/2}.$$

The first term evaluates to 0 at either endpoint, so can be discarded. Meanwhile, the $\sin^2\theta$ factor can be rewritten as $(1-\cos^2\theta)$, giving

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}\cos^{n-2}\theta\cdot\sin\theta\cdot(n-1)\cdot\sin\theta\,\mathrm{d}\theta &= (n-1)\int_{-\pi/2}^{\pi/2}\cos^{n-2}\theta(1-\cos^2\theta)\,\mathrm{d}\theta\\ &= (n-1)(c_{n-2}-c_n).\end{aligned}$$

Rearranging thus gives the equation

$$c_n = \frac{n-1}{n}c_{n-2}.$$

4. This links up to our work in the previous part to give a kind of expression for $\mathrm{vol}(B_1^n)$. Note that $c_2 = \pi/2$ and $c_3 = 4/3$, both of which are strictly less than 2. It follows that in general c_n is strictly less than $\pi/2$ and hence strictly less than 2. In turn, the fraction
$$\frac{\mathrm{vol}(B_1^n)}{\mathrm{vol}(C_1^n)} = \frac{c_n\cdot\mathrm{vol}(B_1^{n-1})}{2\cdot\mathrm{vol}(C_1^{n-1})} < \frac{\pi}{4}\cdot\frac{\mathrm{vol}(B_1^{n-1})}{\mathrm{vol}(C_1^{n-1})}$$
tends to zero, since $\pi/4 < 1$. (ECP)

Math 25b Midterm #2 Solutions

Eric Peterson

Problem 1. 1. Subdivide $[-1,1]\times[-1,1]$ into 4^N subsquares. Produce an estimate for the number of squares that intersect the unit circle.

2. Conclude that the unit circle has measure zero.

Solution. 1. Divide the unit square into four quadrants, restrict attention to the first quadrant, and consider the part of the circle living above the diagonal line $y = x$. Now consider the subdivision of this quadrant into 4^{N-1} subsquares. In each column in the region of interest, we expect the circle to intersect at most two subsquares, since the derivative of $y = \sqrt{1-x^2}$ is bounded in magnitude by 1 in this region. This thus amounts to $2 \cdot 2^N$ subsquares, and together with the other seven half-quadrants we have at most $8 \cdot 2 \cdot 2^N$ subsquares contributing.

2. Since each subsquare has an area 4^{-N}, an upper estimate for the "area" of the circle is $8 \cdot 2 \cdot 2^N \cdot 4^{-N} = 16/2^N$. This tends to zero as N tends to ∞, hence the circle has measure zero. (ECP)

Problem 2. Consider the system of equations

$$\begin{aligned} x + y + \sin(xy) &= h, \\ \sin(x^2 + y) &= 2h. \end{aligned}$$

Does this system have a solution for sufficiently small values $h \in \mathbb{R}$?

Solution. Write the left-hand side as a function:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + \sin(xy) \\ \sin(x^2 + y) \end{pmatrix}.$$

This function is continuously differentiable near the origin, and at the origin we see that

$$D_0 f = \left.\begin{pmatrix} 1 + y\cos(xy) & 1 + x\cos(xy) \\ 2x\cos(x^2+y) & \cos(x^2+y) \end{pmatrix}\right|_{(x,y)=(0,0)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is an invertible operator. Hence, the inverse function theorem applies, so that f admits an inverse near the origin. As a special case, the domain of the local inverse includes values like $(h, 2h)$ for $h \ll 1$, and hence solutions are guaranteed to exist. (ECP)

Problem 3. Let A and B be Jordan-measurable subsets of $\mathbb{R}^3$. For every $c \in \mathbb{R}$, let $A_c = \{(x, y) : (x, y, c) \in A\}$ and define B_c similarly. Suppose each A_c and B_c are Jordan-measurable and have the same area. Show that A and B have the same volume.

Solution. Since A, B are Jordan-measurable, they are bounded and have integrable characteristic functions.

Let $R = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ be a rectangle containing both A and B.

Then we can write the volumes of A and B as $\int_R \chi_A$ and $\int_R \chi_B$ respectively.

By Fubini's theorem, these are equal to

$$\int_{[a_1,b_1]\times[a_2,b_2]} \int_{[a_3,b_3]} \chi_A dcd(x,y)$$

and

$$\int_{[a_1,b_1]\times[a_2,b_2]} \int_{[a_3,b_3]} \chi_B dcd(x,y)$$

respectively.

Using the fact that the slices A_c and B_c have the same area, we calculate

$$\begin{aligned}\int_{[a_1,b_1]\times[a_2,b_2]} \int_{[a_3,b_3]} \chi_A dcd(x,y) &= \int_{[a_1,b_1]\times[a_2,b_2]} \left(\int_{[a_3,b_3]} \chi_{A_c} dc\right) d(x,y) \\ &= \int_{[a_1,b_1]\times[a_2,b_2]} \left(\int_{[a_3,b_3]} \chi_{B_c} dc\right) d(x,y).\end{aligned}$$

Therefore, A and B have the same volume. (RP)

Problem 4. Consider the space of matrices

$$\mathbb{R}^4 \xrightarrow{\cong} M_{2\times 2},$$
$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

1. Show that the zero matrix is the only singular point of $\det\colon M_{2\times 2} \to \mathbb{R}$.
2. Describe $P^2_{\det @ 0}(a, b, c, d)$. Calculate the signature of the quadratic part and use it to classify the critical point at 0.
3. (Bonus:) Give a reason why you might have guessed the answer to part 1 without any computation.

Solution. 1. The partial derivatives of det assemble into the matrix

$$D_A \det = \begin{pmatrix} d & -c & -b & a \end{pmatrix}.$$

This matrix vanishes exactly if $a = 0$, $b = 0$, $c = 0$, and $d = 0$ are all satisfied.

2. The function $\det(A) = ad - bc$ is itself a quadratic function, hence is its own second degree Taylor polynomial. More than this, there is no constant term at 0, so it is even its own quadratic form. As mentioned repeatedly in class, for an isolated mixed term like ad we can find variables $a = u - v$, $d = u + v$ so that the expansion has no mixed terms, resulting in
$$ad - bc = u^2 - v^2 + w^2 - t^2.$$
The signature of this form is $(2, -2)$, hence is a saddle.

3. The derivative is meant to encode how the value of the function changes when moving along a tangent direction. For any nonsingular matrix, there is a direction one can move to make the matrix "more singular" by bringing the columns closer together, hence the derivative decreases. For a nonzero singular matrix, there is a direction one can move (orthogonal to the output of the matrix) which will cause the matrix to no longer be singular and hence det to be nonzero. Lastly, right at the zero matrix, no single direction of change in the matrix will cause the matrix to be nonsingular, hence the derivative is zero. (ECP)

Problem 5. Use Fubini's theorem to compute the integral
$$\int_0^\pi \int_y^\pi \frac{\sin x}{x} \, dx \, dy.$$

Solution. As instructed, we use Fubini's theorem to present the integral another way:
$$\int_0^\pi \int_y^\pi \frac{\sin x}{x} \, dx \, dy = \int_0^\pi \int_0^x \frac{\sin x}{x} \, dy \, dx = \int_0^\pi (x - 0) \frac{\sin x}{x} \, dx = 2.$$

(Though the problem didn't ask you to do so, if you wanted to check that $\sin x/x$ satisfied the hypotheses of Fubini's theorem, you would note that the function is continuous away from $(0, y)$, and there it is *defined* by continuous extension, using the nontrivial single-variable fact that $\lim_{x\to 0} \sin x/x = 1$.) (ECP)